

Quenched Asymptotics for the Discrete Fourier Transforms of a Stationary Process

David Barrera

Department of Mathematical Sciences
University of Cincinnati
PO Box 210025, Cincinnati, Oh 45221-0025, USA.
barrerjd@mail.uc.edu

Bernarda y Gonzalo

Abstract

In this dissertation, we show that the Central Limit Theorem and the Invariance Principle for Discrete Fourier Transforms discovered by Peligrad and Wu can be extended to the quenched setting. We show that the random normalization introduced to extend these results is necessary and we discuss its meaning. We also show the validity of the quenched Invariance Principle for fixed frequencies under some conditions of weak dependence. In particular, we show that this result holds in the martingale case.

The discussion needed for the proofs allows us to show some general facts apparently not noticed before in the theory of convergence in distribution. In particular, we show that in the case of separable metric spaces the set of test functions in the Portmanteau theorem can be reduced to a countable one, which implies that the notion of quenched convergence, given in terms of convergence a.s. of conditional expectations, specializes in the right way in the regular case when the state space is metrizable and second-countable.

We also collect and organize several disperse facts from the existing theory in a consistent manner towards the statistical spectral analysis of the Discrete Fourier Transforms, providing a comprehensive introduction to topics in this theory that apparently have not been systematically addressed in a self-contained way by previous references.

Acknowledgements

Thanks to M. Peligrad for introducing me to the problems studied here and guiding my steps through the techniques that made possible to write this monograph. She has been the ideal Ph.D. advisor to me: confident, approachable, generous, and capable of mixing the demands for “readings and deadlines” with the permission for “wondering freedom” in amounts that make possible the emergence of creative ideas.

I would like to extend my gratitude to W. Bryc and Y. Wang, the members of the evaluation committee for my Ph.D. dissertation, for their criticism and suggestions to improve the work here presented and for their academic advise.

My conversations with C.Dragan were particularly useful and stimulating, and D.Volný personally encouraged me to work on the details of the proof of Theorem 16.3, pointing out the fact that his proof (with M.Woodroffe) on the non-rotated case was applicable to the discrete Fourier transforms.

This work was partially supported by the Research Grant no.1512936 from the Division of Mathematical Sciences of the National Science Foundation.

Contents

Introduction	1
I Background and Results	7
1 Background Theory	9
1 The Koopman Operator and its Point Spectrum	10
1.1 Definitions and General Properties	10
1.2 Separability and Cardinality of the Point Spectrum	13
2 Random Elements in L^2	13
2.1 Functions Defined by Limits	14
2.2 The Fourier Transform of an Integrable Function	15
2.3 A Duality Theorem	17
2.4 Duality via Decay of Second Moments	19
3 Dunford-Schwartz Operators and the Ergodic Theorem	21
3.1 The Ergodic Theorem for Positive Dunford-Schwartz Operators . .	21
3.2 The Ergodic Theorem for Discrete Fourier Transforms	23
3.3 Dunford-Schwartz Operators and the Weak L^p -spaces	26
4 T -Filtrations and Adapted Processes	27
4.1 Definitions and Examples	27
4.2 Heuristic Interpretation	30
4.3 Interactions with the Koopman Operator	30
5 The Autocovariance Function and the Spectral Density	32
5.1 The Autocovariance Function	33
5.2 The Spectral Density	34
5.3 Regular Processes	35
5.4 On the Existence of the Spectral Density	36
5.5 Estimating the Spectral Density	41
2 Convergence in Distribution	45
6 A Refinement of the Portmanteau Theorem	46
7 Convergence of Complex-valued Cadlag Functions	47
7.1 The topology of $D[[0, \infty), \mathbb{C}]$	48
7.2 Convergence on $D[[0, \infty), \mathbb{C}]$	49

8	Convergence of Types	51
8.1	Preliminary Facts	52
8.2	Convergence of Types Results	52
9	Random Elements and Product Spaces	53
10	A Transport Theorem	55
3	Quenched Convergence and Regular Conditional Expectations	57
11	Definitions and General Remarks	58
11.1	Quenched Convergence	58
11.2	Regular Conditional Expectations	60
11.3	Regularity and T -Filtrations	62
12	Examples of Regularity	63
12.1	Functions of i.i.d. Sequences	64
12.2	Functions of Stationary Markov Chains	65
13	Regularity and Quenched Convergence	72
14	Product Spaces and Regularity	73
4	Quenched Asymptotics of Normalized Fourier Averages	75
15	The quenched CLT for Fourier Transforms	76
16	The Random Centering	79
16.1	Necessity of the Random Centering	79
16.2	Cases of Quenched Convergence without Random Centering	81
17	Quenched Functional Central Limit Theorem	82
17.1	The Invariance Principle for Averaged Frequencies	83
17.2	Invariance Principles for Almost Every Fixed Frequencies	84
II	Proofs	87
5	Martingale Case	91
18	Preliminary Results	91
19	Martingale Case	93
6	Proofs of Theorems 15.1, 17.1 and 17.2	99
20	Martingale Approximations	100
20.1	Approximation Lemmas	100
20.2	The Approximating Martingales	104
21	Proof of Theorem 15.1	105
22	Proof of Theorems 17.1 and 17.2	107
22.1	Proof of Theorem 17.1	109
22.2	Proof of Theorem 17.2	111
22.3	A Note on Theorem 10.1	115
7	Proofs Related to the Random Centering	117
23	Proof of Theorem 16.1	118
24	Theorem 15.1 for Adapted Linear Processes	120
25	Proof of Theorem 16.3	122

26	Proof of Theorems 16.5 and 16.6	126
26.1	Proof of Theorem 16.5	126
26.2	Proof of Theorem 16.6	127

Notation

1. *The natural numbers.* We will denote by \mathbb{N} the set of natural numbers starting at zero. $\mathbb{N} := \{0, 1, 2, \dots\}$. We will also use the notation $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.
2. *The space $([0, 2\pi), \mathcal{B}, \lambda)$.* Throughout this text, $([0, 2\pi), \mathcal{B}, \lambda)$ will denote, unless otherwise specified, the interval $[0, 2\pi)$ seen as probability space with the Borel sigma-algebra \mathcal{B} and the normalized Lebesgue measure λ . This is, for every $B \in \mathcal{B}$

$$\lambda(B) = \frac{1}{2\pi} L(B) \quad (1)$$

where L is the Lebesgue measure, specified by $L[a, b) = b - a$ for every real numbers $a < b$.

3. *Limits.* Unless otherwise specified, an expression of the form “ \lim_n ” must be read as “ $\lim_{n \rightarrow \infty}$ ”, and similarly for “ \limsup_n ” and “ \liminf_n ”.
4. *Convergence of series.* Given a sequence $(a_k)_{k \in \mathbb{Z}}$ of elements in a normed vector space V , we say that $\sum_{k \in \mathbb{Z}} a_k$ is convergent if $\sum_{k \in \mathbb{N}} a_k$ and $\sum_{k \in \mathbb{N}^*} a_{-k}$ are convergent (the partial sums have a limit), and we define $\sum_{k \in \mathbb{Z}} a_k := \sum_{k \in \mathbb{N}} a_k + \sum_{k \in \mathbb{N}^*} a_{-k}$.
5. *Measurability.* Given measurable spaces (Ω_1, \mathcal{F}) , (Ω_2, \mathcal{G}) , a function $f : \Omega_1 \rightarrow \Omega_2$ is \mathcal{F}/\mathcal{G} measurable if for every $B \in \mathcal{G}$, $f^{-1}(B) \in \mathcal{F}$. If (Ω_2, \mathcal{G}) is the space of complex numbers with the Borel sigma algebra \mathcal{C} , we will use the term \mathcal{F} -measurable function when referring to an \mathcal{F}/\mathcal{C} measurable function. For a specified \mathcal{F} , clear along the discussion, we will speak of a measurable function when referring to an \mathcal{F} -measurable function.
6. *Preimages of sets.* Given measurable spaces (Ω, \mathcal{F}) and (Ω', \mathcal{F}') , $A \in \mathcal{F}'$ and an \mathcal{F}/\mathcal{F}' measurable function X , we will denote by $[X \in A]$ the \mathcal{F} -set

$$[X \in A] := \{\omega \in \Omega : X(\omega) \in A\}.$$

7. *Equivalence classes of functions.* If $(\Omega, \mathcal{F}, \mu)$ is a measure space and (Ω', \mathcal{F}') is a measurable space, we say that two \mathcal{F}/\mathcal{F}' -measurable functions X, Y are μ -equivalent if there exists $A \in \mathcal{F}$ with $\mu(\Omega \setminus A) = 0$ such that $X(\omega) = Y(\omega)$ for every $\omega \in A$. If μ is fixed and X is μ -equivalent to Y , we call X a version of Y .
8. *Two abbreviations.* Here, “a.s.” abbreviates “almost surely”, and “a.e” abbreviates “almost every” (not “almost everywhere”).
9. *L^p spaces.* Given a measure space $(\Omega, \mathcal{F}, \mu)$, a sigma algebra $\mathcal{F}_0 \subset \mathcal{F}$, and $0 < p < \infty$, $L^p_\mu(\mathcal{F}_0)$ denotes the (normed) space of μ -equivalence classes of p -integrable functions $X : \Omega \rightarrow \mathbb{C}$ that are \mathcal{F}_0 -measurable (with the norm given in the next item). Thus, $X \in L^p_\mu(\mathcal{F}_0)$ if and only if (some version of) X is \mathcal{F}_0 -measurable and

$$\int_\Omega |X(\omega)|^p d\mu(\omega) < \infty.$$

If \mathcal{F} is fixed, we will use the notation L_μ^p for $L_\mu^p(\mathcal{F})$. $L_\mu^\infty(\mathcal{F})$ denotes the (normed) space of μ -equivalence classes of essentially bounded functions: $X \in L_\mu^\infty(\mathcal{F})$ if there exists $c > 0$ such that

$$\mu(|X| > c) = 0.$$

10. *L^p norms.* Given $p > 0$ and $X \in L_\mu^p$, “ $\|X\|_{\mu,p}$ ” will denote the L^p -norm of X . This is

$$\|X\|_{\mu,p} := \left(\int_\Omega |X(\omega)|^p d\mu(\omega) \right)^{1/p} \quad (2)$$

when $p < \infty$, and

$$\|X\|_{\mu,\infty} := \inf\{c > 0 : \mu(|X| > c) = 0\} \quad (3)$$

when $p = \infty$.

11. *The spaces $l^p(\mathbb{Z})$ and $l^p(\mathbb{N})$.* If $\Omega = \mathbb{Z}$ or $\Omega = \mathbb{N}$ and μ is the counting measure ($\mu(\{z\}) = 1$ for every $z \in \Omega$), we will denote by $l^p(\mathbb{Z})$ (resp. $l^p(\mathbb{N})$) the space L_μ^p . Thus $(a_k)_k$ ($k \in \mathbb{Z}$ or \mathbb{N}) belongs to $l^p(\mathbb{Z})$ (resp. $l^p(\mathbb{N})$) if and only if

$$\|(a_k)_k\|_{\mu,p}^p = \sum_k |a_k|^p < \infty. \quad (4)$$

12. *Random variables and stochastic processes.* A *random variable* is a \mathbb{P} -equivalence class of measurable functions $X : \Omega \rightarrow \mathbb{C}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (note that here random variables are complex valued functions). A *stochastic process* is a sequence $(X_k)_k$ of random variables, where k runs over \mathbb{Z} or \mathbb{N} .
13. *Convergence in distribution.* The convergence in distribution of random elements in a metric space (or of probability measures, or of distribution functions) will be denoted here by “ \Rightarrow ”. If necessary, we will use the notation “ \Rightarrow_n ” to indicate that the convergence holds as $n \rightarrow \infty$.
14. *Characteristic functions.* Given a measurable space (Ω, \mathcal{F}) and $A \in \mathcal{F}$, we will use the notation I_A for the *characteristic function* of A . This is $I_A : \Omega \rightarrow \{0, 1\}$ is given by $I_A(\omega) = 0$ if $\omega \notin A$ and $I_A(\omega) = 1$ if $\omega \in A$.
15. *Expectation.* If \mathbb{P} is a probability measure other than λ , we will use the traditional notation “ E ” to denote integration with respect to \mathbb{P} . Thus for instance $\|X\|_{\mathbb{P},1} = E[|X|]$ if $X \in L_\mathbb{P}^1$. If we need to specify \mathbb{P} , we will use the notation “ $E_\mathbb{P}$ ”, or some other convenient variation of it, to indicate integration with respect to \mathbb{P} .
16. *Inner Product in L^2 .* We will also make use of the Hilbert space structure of L_μ^2 , whose inner product $\langle X, Y \rangle_\mu : L_\mu^2 \times L_\mu^2 \rightarrow [0, \infty)$ is defined by

$$\langle X, Y \rangle_\mu = \int_\Omega X(\omega) \overline{Y}(\omega) d\mu(\omega), \quad (5)$$

where $\overline{Y}(\omega)$ is the conjugate of $Y(\omega)$, and we will say that $X, Y \in L_\mu^2$ are *orthogonal* if $\langle X, Y \rangle_\mu = 0$.

17. *The one-dimensional torus.* Finally, $\mathbb{T} \subset \mathbb{C}$ denotes the unit circle with the subspace topology and the Lie-group structure given by multiplication of complex numbers.

Introduction

The celebrated *Birkhoff's Ergodic Theorem* states that if $T : \Omega \rightarrow \Omega$ is a measure-preserving transformation on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $X \in L^1_{\mathbb{P}}(\mathcal{F})$, and $X_k := X \circ T^k$, then the ergodic averages

$$A_n = \frac{1}{n} \sum_{k=0}^{n-1} X_k$$

converge \mathbb{P} -a.s., as $n \rightarrow \infty$, to a function \hat{X} with $\hat{X} \circ T = \hat{X}$ (\mathbb{P} -a.s.).

A well-known and easy argument¹ allows one to see that Birkhoff's Ergodic Theorem “generalizes itself” in the following way: let

$$S_n(\theta) := \sum_{k=0}^{n-1} X_k e^{ik\theta}$$

be the n -th discrete Fourier Transform of the process $(X_k)_k$. Then the Fourier averages

$$A_n(\theta) := \frac{1}{n} S_n(\theta)$$

converge \mathbb{P} -a.s., as $n \rightarrow \infty$. This is (a partial statement of) the *pointwise Ergodic Theorem for Discrete Fourier Transforms*, a version of Birkhoff's Ergodic Theorem whose further analysis has taken mainly the directions opened by the following questions:

Question 1: *Given a version of X . Can we choose the (probability one) set of convergence for $A_n(\theta)$ independent of θ ?*

The answer to this particular question appeared 1941 when Wiener and Wintner found a positive answer² known today as the *Wiener-Wintner Theorem*, a result that opened a line of research that would lead to results such as Bourgain's *Return Times Theorem* ([13]) and to the currently very active investigation of convergence theorems for multiple recurrence in Dynamical Systems. These investigations lie at the heart of the connections

¹See Theorem 3.2 and its proof.

²Though, according to Assani ([2], p.24), Wiener and Wintner's original proof was flawed, and the first known correct proof of the Wiener-Wintner Theorem is actually due to Furstenberg ([2], p.36), who published it in his 1960's monograph [28].

between probabilistic or ergodic-theoretical techniques and problems in number theory, like Furstenberg’s equivalence (and proof) of Szemerédi’s theorem ([29]).

Question 2: *What can be said about the asymptotics of the periodogram*

$$I_n(\theta) = \frac{|S_n(\theta)|^2}{n} \quad ? \quad (6)$$

The importance of this question came mainly from the research in the direction of the *Periodogram Analysis* (or, more widely, *Spectral Analysis*) of Time Series, a technique started by Schuster in 1898 ([44]) that would become the standard tool for the identification of statistically significant frequencies in time series of observations and has been widely applied in the Physical and Social Sciences. Several papers appeared through the 20th and 21st century addressing this and related questions in different cases important for the applications³, some of them departing from the elementary fact that the periodogram is the square of the modulus of $\sqrt{n}A_n(\theta)$, and therefore that the investigation of the Periodogram’s asymptotics can be seen as a particular instance of the question about the speed of convergence of $A_n(\theta)$.

The Central Limit Theorem for Discrete Fourier Transforms

The 2010 paper [41] by Peligrad and Wu, devoted to the asymptotics of $\sqrt{n}A_n(\theta)$, is a remarkable step in this direction of the research on Spectral Analysis. It is shown there that, if we take into account a certain T -filtration⁴ associated to the process $(X_k)_k$, the assumptions necessary to prove the Central Limit Theorem for $A_n(\theta)$ -this is, that $\sqrt{n}A_n(\theta)$ is asymptotically normal- can be basically reduced to the minimal ones plus a certain regularity condition (see Definition 5.4)⁵. The results on that paper contain many of the precedent ones as special cases. They are also stated in the setting typical for the investigation of quenched limit theorems and gave rise to the main questions addressed in this monograph.

Without going now into details, it is important to notice that, in contrast to the \mathbb{P} -a.s. convergence of $A_n(\theta)$, to pass from asymptotic results for $\sqrt{n}A_n$ to asymptotic results for (the complex-valued) process $\sqrt{n}A_n(\theta)$, a further analysis of the joint distributions of its real and imaginary parts is needed. Upon addressing this problem, one realizes that the “frequencies” (values of θ) associated to the “square root” of the *point spectrum* of the Koopman (composition) operator induced by the map T (definitions 1.2 and 1.3) have the remarkable property of being the “generic” set of exceptional frequencies in which the asymptotics of $\sqrt{n}A_n(\theta)$ can fail to be (2-dimensional) normal with independent entries. For this and other reasons, the point spectrum of the Koopman operator will play an essential role in the results to be presented here, and the exposition starts with the basic definitions and properties related to it.

³For a review of some of them see the introduction to [41] and [47] and the references therein.

⁴See definitions 4.2 and 4.3 in this monograph.

⁵As the reader will see, such condition is actually unnecessary in the quenched setting, because of the “random centering” needed for the corresponding results.

The Invariance Principle for Discrete Fourier Transforms

One can roughly summarize Peligrad and Wu's Central Limit Theorem by saying that, under regularity, the distribution of $\sqrt{n}A_n(\theta)$ is (indeed) asymptotically normal with independent real and imaginary parts for λ -a.e. fixed frequency θ . Peligrad and Wu's paper addresses also the problem of the invariance principle but, in contrast to the case corresponding to the Central Limit Theorem, the authors show the weaker statement that the asymptotic distribution of $W_n(\theta, t) := \sqrt{n}A_{[nt]}(\theta)$ corresponds to that of a random function of the form $(\theta, \omega) \mapsto f(\theta)(B_1(\omega) + iB_2(\omega))$ (with *random* parameters ω and θ) where B_1 and B_2 are independent Brownian motions⁶. The underlying probability law is therefore $(\lambda \times \mathbb{P}) W_n^{-1}$: the parameter θ is considered only "in average" in this case.

The bottom line of the problem when trying to prove the Invariance Principle for fixed frequencies with these methods lies in the lack of a maximal inequality general enough as to pass from the martingale approximations for $\sqrt{n}A_n(\theta)$ to martingale approximations for $t \mapsto \sqrt{n}A_{[nt]}(\theta)$. On the other side, integrating over θ allows us to apply Hunt and Young's inequality (Theorem 2.3 in this monograph), which actually has a role in the fixed frequency approximations, to bypass this problem. To the date, it is not known whether the Central Limit Theorem of Peligrad and Wu for fixed frequencies can be extended to a corresponding invariance principle without additional assumptions.

The Problem of Quenched Convergence

The results listed before are stated in the context of *stationary* sequences. There is a certain form of non-stationarity that is very important in the applications and has grown as a topic of intensive research during the last twenty years. In the context of i.i.d. sequences it can be introduced in the following way⁷: let $(\zeta_k)_{k \in \mathbb{Z}}$ be an i.i.d. sequence, let $f = f(\dots, z_{-1}, z_0)$ be a measurable real-valued function defined on the space of complex-valued sequences indexed by the non-positive integers with the product sigma algebra, and consider the (stationary) stochastic process $(X_k)_k$ given by

$$X_k := f(\dots, \zeta_{k-1}, \zeta_k)$$

for all $k \in \mathbb{N}$.

Assume, for the sake of the discussion, that $EX_0 = 0$ and $EX_0^2 = 1$, and that we have proved the Central Limit Theorem for the stationary process $(X_k)_k$, so that

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} X_k \Rightarrow_n N(0, 1).$$

Question (A question on Quenched Convergence.) *If we fix a point $a = (\dots, a_{-1}, a_0)$ in the domain of f and consider the (nonstationary) process $(X_{a,k})_k$ given by*

$$X_{a,k} = f(\dots, a_{-1}, a_0, \zeta_1, \dots, \zeta_k),$$

⁶Actually $f(\theta) = \sigma(\theta)/\sqrt{2}$, where $\theta \mapsto \sigma^2(\theta)$ is the *spectral density* of $(X_k)_k$ with respect to the normalized Lebesgue measure. See Section 5 for details.

⁷The formal definition is Definition 11.1 in page 58.

does the “same” Central Limit Theorem (still) hold for $(X_{a,k})_k$?

This is, can we assert that

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} X_{a,k} \Rightarrow_n N(0, 1) \quad ?$$

The idea behind the notion of quenched convergence is whether we can give asymptotics for a stochastic process “started at a point”, or with “initial conditions”. In this particular example “quenched convergence” means an affirmative answer to the question above for almost every (\dots, a_{-1}, a_0) with respect to the law of $(\dots, \zeta_{-1}, \zeta_0)$.

Note that the law of the modified process $(X_{a,k})_k$ is typically singular with respect to the law of the stationary process $(X_k)_k$ (for instance $\mathbb{P}[X_{a,0} = f(a)] = 1$ but $\mathbb{P}[X_0 = f(a)]$ is typically equal to zero), and therefore we cannot give affirmative answers to the question above based on arguments of dominating measures.⁸

The formal notion of quenched convergence, which captures the question above and other versions of it, is actually strictly stronger than the notion of convergence in distribution⁹: every process converging in the quenched sense converges in “the annealed” sense, but the reciprocal is not true, even in the specific setting of the question above. This will be stated “abstractly” in Section 16 and proved (in the setting of functions of i.i.d. sequences, and for the corresponding normalized Fourier averages $\sqrt{n}A_n(\theta)$) in Chapter 7. Note also that this is not obvious: consider for instance the $(m+1)$ -dependent case in which f has “finite memory”, $f = f(z_{-m}, \dots, z_0)$ for some $m > 0$.¹⁰

A Growing Trend

The problem of quenched convergence was not intensively studied during the 20th century, though it has been long recognized as an important requirement in the theory of statistical inference for Markov processes¹¹.

Results on quenched convergence can be traced to at least 1968 with Billingsley’s quenched Invariance Principle for ϕ -mixing processes ([9], Theorem 20.4). Other results in this direction appeared sporadically¹², but an inflexion point came with the paper [24] published in 2001 by Derrienic and Lin, which was inspired by a question raised by Kipnis and Varadhan in 1987 ([34], Remark 1.7) and gave rise to a considerable amount of new research¹³ on the validity of the Central Limit Theorem for functions of Markov chains

⁸See [10], Theorem 14.2 for an example of this technique.

⁹See Remark 11.1 in page 59.

¹⁰Or, in a more strict language, $f((z_{-k})_{k \in \mathbb{N}}) = f((z'_{-k})_{k \in \mathbb{N}})$ for any two sequences whose terms coincide for $0 \leq k \leq m$.

¹¹See for instance the note preceding (1.8) in [12]. See also Example 7 in page 71 in this monograph for technical details on the relationship between quenched convergence -as presented here- and convergence with respect to the transition measures induced by the Kernel of a stationary Markov chain.

¹²According to the remarks in [19], the paper [31] deserves special mention in this respect, since it started the investigation of these results in the sense of Markov operators. See also [6] for a slightly more detailed account of the results in this direction before 2001.

¹³In order of appearance, some examples are [48], [17], [39], [20], [21], [18], [45], [46], [5], and [6].

when the chain starts at a point. In a more informal way it can be asserted that, nowadays, the word “quenched” is becoming a common sound in the conferences and meetings of specialists in Probability.

The Content of this Monograph

This work presents the first series of results on quenched limit theorems for the discrete Fourier transforms of a stationary process.

While the original purpose was to limit the exposition to the minimal amount of material necessary to fully understand the results presented in the series of papers [3], [4], and [5], and therefore to refer the reader to the existing literature for the background theory, I found particularly difficult to navigate between the many references needed to carry on the proofs of the results in question while maintaining at the same time a clear perspective of the mathematical ground in which these arguments rest. For that reason, a chapter on “Background Theory”, Chapter 1, was inserted. While it is my desire that it can serve as a quick introduction for anyone interested in reading the series of papers started by [47] in the direction of the speed of convergence for the discrete Fourier transforms, the results presented in this chapter are not original, and my motivation to present them was to pave the way to a clear exposition in further sections. I have tried to keep the references to the literature containing the original proofs even in the cases in which, for pedagogical reasons, I decided to rewrite them. This was not always possible though, and I must advance my apologies to any reader who finds a proof by a different author without a reference, expecting that (s)he believes in the unintentional nature of my omission.

Chapter 2 covers issues related to convergence in distribution. The following reasons lead me to insert these topics as part of this monograph: first, although perhaps obvious for the expert that knows the real-valued case, the notion of convergence in distribution for *complex-valued* cadlag functions is not easy to find in the mainstream literature, thus I decided that it was wise to devote a few pages explaining how this notion can be understood via an obvious extension of the Skorohod metric to the complex-valued case, and how the techniques used for real-valued functions *indeed* apply to the complex-valued ones. For the same reason, I also considered important to explain why some well-known convergence of types theorems can be carried over to the complex-valued case, and to give them as statements of convergence of random variables instead of distribution functions. The “transport theorem” in Section 10, borrowed from an external source, was inserted in order to make the monograph more self-contained.

The “refinement of the Portmanteau theorem” (Section 6) deserves, on the other side, special mention. It came out after many hours confronting a certain question that has some resemblance to the one giving rise to the Wiener-Wintner theorem: when facing the problem of passing from the “fixed frequency” to the “averaged frequency” limit theorems, which in the annealed case can be trivially solved by integrating with respect to the parameter θ , one has to deal with the fact that, in the quenched case, the (probability one) set of decomposing measures with respect to which the results hold for a fixed θ may change with θ , and therefore one has to be more careful when performing integrations over the (uncountable) set $[0, 2\pi)$ of parameters θ . While this can be done via arguments

involving interpretations of Fubini's theorem, I found more illuminating and clear to use the language of conditional expectations in this case, but in order to succeed with this way one is finally lead to ask whether the set of test functions in the Portmanteau theorem can be reduced to a countable one. The answer is “yes” in the separable case (and it is what this “refinement” deals with), and the consequences for the theory of quenched convergence pay off, in my opinion, the short digression.

Chapter 3 presents the definitions and elementary properties related to the notion of quenched convergence: adapted T -filtrations, regular conditional expectations, and the interactions between the product measures and the regular conditional expectation with respect to the product of two sigma algebras. These notions constitute the elementary “grammar” necessary for the results on quenched convergence presented here and for their proofs, and are usually taken for granted along the papers in the literature. For this reason, this is also a chapter aimed to introduce the beginner to these techniques.

This chapter presents also several examples related to the existence of regular conditional expectations. For the most part, they belong to the standard literature, but since some of the constructions along the references are not given in terms of invertible Dynamical Systems, I considered appropriate to spend some energy explaining how the corresponding results are indeed possible if we restrict ourselves to the invertible case. In particular, we obtain a representation of a stationary stochastic process as a sequence of functions of a stationary Markov Chain preserving the invertibility of the underlying shift operator. The construction can be easily adapted to show that any stochastic process admits a representation as a function of a (possibly nonstationary) Markov Chain.

The main contributions of this monograph are contained in Chapter 4. In summary, it is shown there that the limit theorems by Peligrad and Wu admit quenched versions under some “intuitively obvious” modification (the “random centering”). It is shown that this modification is necessary, and some quenched invariance principles for fixed frequencies are also provided.

The rest of the monograph is devoted to prove the results in Chapter 4. Since it is not possible to make comments about this without going into technical details we will just mention two things:

First, the reader is invited to note that, in a certain sense, all the quenched results given here for fixed frequencies are just interpretations of corresponding results for (non-rotated) partial sums (including, after a “creative” step, the proof of Theorem 15.1), and therefore we can consider the investigation of asymptotics for the discrete Fourier transforms $S_n(\theta)$ for θ fixed (almost) as a particular case of the investigation of the asymptotics for partial sums of (complex-valued) random variables.

Second. For the reader familiar with these techniques, note that the standard application of maximal inequalities to pass from martingale approximations leading to the Central Limit Theorem to corresponding approximations leading to the Invariance Principle encounters an additional obstacle here: in the nonstationary setting, maximal inequalities are scarce. This is the bottom line behind the necessity of weak- L^p spaces along our proofs, and it provides a further reason to call for the (already growing) investigation of maximal inequalities for nonstationary processes.

Part I

Background and Results

Chapter 1

Background Theory

In this chapter we will survey the background theory necessary to justify our forthcoming discussions and to settle a solid ground for them. Specifically, we will be concerned with presenting the objects that motivate the questions leading to the main results in this monograph, leaving aside for later chapters the discussions relative to the methods of our proofs.

Most of the results presented in this chapter are part of the literature and the reader is referred to the corresponding reference for their proofs. Nonetheless, we will go through detailed discussions whenever the clarity of the arguments would be affected otherwise.

This chapter is organized as follows: in Section 1 we discuss the notion of the *Koopman operator* (Definition 1.2) associated to a measure preserving transformation, emphasizing the discussion on the structure of its point spectrum. These notions will show up later along the proofs of our main results, particularly in the steps involving asymptotic finite-dimensional distributions.

Then, in Section 2, we will present some results necessary to clarify the construction of the approximating martingales whose asymptotics will be transferred to the processes under consideration. This will require a short review of results from classical Harmonic Analysis and a visit to the problem of measurability for functions defined by limits.

Section 3 presents a result (Theorem 3.1) that seems to be implicit in the literature but whose pieces are somehow disperse. This theorem gives rise to a result (Theorem 3.2) that generalizes the pointwise and L^p ergodic theorems to discrete Fourier transforms in a very natural way, justifying the investigation of its rate of convergence via the Central Limit Theorem. To reach this result we have to introduce a technical notion, the “*extension to the product space*” of a random variable and a measure-preserving transformation (see Definition 3.3 and the discussion following it), that will be important for some of the steps in the forthcoming proofs of our main results. We also introduce the basics of *weak L^p -spaces*, which will be needed later along the proofs from Chapter 4.

In Section 4 we settle the ground for the forthcoming discussions about “quenched con-

vergence". In particular, we will establish (Proposition 4.1) the interaction between the Koopman operator and the conditional expectations with respect to the corresponding elements in the filtration of an adapted process, a fact that will be crucial for our proofs. We will also introduce important notions such as that of a (strictly) "stationary process" (Definition 4.1), "left" and "right" sigma-algebras (Definition 4.4), and the "model" example of linear processes (Example 1). We conclude with two ergodic theorems (Theorem 4.1 and Corollary 4.2) that will be of utter importance when discussing the quenched asymptotic distributions associated to the normalized Fourier averages, and in particular to understand the role of the point spectrum in the statements of the results to be presented in Chapter 4.

Finally, in Section 5, we will introduce the notion of the *autocovariance function* (Definition 5.2) and the *spectral density* (Definition 5.3) of a stationary square-integrable process, whose estimation justify much of the research in the directions explored along this monograph. We will also introduce the notion of *regular processes* (Definition 5.4), which will be essential for some of our proofs. Our discussion will lead us to the (annealed) limit theorems of Peligrad and Wu (theorems 5.5 and 5.6), whose extension to the quenched setting is one of the main purposes of this work.

1 The Koopman Operator and its Point Spectrum

In this section we present the notion of the *Koopman operator* associated to a measure preserving transformation on a probability space, and we introduce the analytic facts about it that will be of use along the proofs of the results present in this monograph.

1.1 Definitions and General Properties

Let us start by recalling the notion of a *measure preserving transformation*.

Definition 1.1 (Measure Preserving Transformation). *Given a measure space $(\Omega, \mathcal{F}, \mu)$, a **measure preserving transformation** $T : \Omega \rightarrow \Omega$ is an \mathcal{F}/\mathcal{F} -measurable map such that for every $A \in \mathcal{F}$*

$$\mu(T^{-1}A) = \mu(A). \quad (1.1)$$

We will restrict our attention in this monograph to measure-preserving transformations on probability spaces, but some of the notions presented below can be extended to more general measure spaces.

In particular, measure preserving transformations and their dynamics will be of utter importance to codify the notion of stationary processes used along this work (see Definition 4.1 below). To settle the ground for the upcoming discussions let us introduce now the notion of the *Koopman operator* associated to a measure preserving transformation.

Definition 1.2 (Koopman Operator). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $T : \Omega \rightarrow \Omega$ be a measure preserving transformation. Given $p > 0$ we define the **Koopman operator** $T : L_{\mathbb{P}}^p \rightarrow L_{\mathbb{P}}^p$ by*

$$TY := Y \circ T.$$

Remark 1.1. Note that we are using the same notation for the transformation T and its associated Koopman operator. This should not be a source of confusion in what follows: “ TU ” must be interpreted as $U \circ T$ when U is a random variable, and as the image of U under T when U is a subset of Ω . Similarly, $T^{-1}U$ should be understood as $U \circ T^{-1}$ if U is a random variable when T is invertible and bimeasurable, and as $T^{-1}(U)$, the inverse image of U under T , if U is a subset of Ω .

Since many of our forthcoming proofs depend on spectral properties of the Koopman operator associated to a measure-preserving transformation, we will start by presenting some elementary facts related to the eigenvalues of these operators. Let us start by a formal introduction of these objects.

Definition 1.3 (Point Spectrum of T). *With the notation in Definition 1.2, denote by*

$$\text{Spec}_p(T) := \{\alpha \in \mathbb{C} : \text{there exists } q > 0 \text{ and } X \in L_{\mathbb{P}}^q \setminus \{0\} \text{ with } TX = \alpha X\}.$$

*$\text{Spec}_p(T)$ is called the **point spectrum** of T , and any element of $\text{Spec}_p(T)$ is called an **eigenvalue** of T .*

Remark 1.2. Note that if $p > 0$, T is an isometry in $L_{\mathbb{P}}^p$: $(E[|TX|^p])^{1/p} = (E[|X|^p])^{1/p}$. In particular, $\text{Spec}_p(T) \subset \mathbb{T}$.

The following proposition shows that the definition of $\text{Spec}_p(T)$ can be recast by restricting T to $L_{\mathbb{P}}^q$ for a fixed $q > 0$.

Proposition 1.1 (Persistence of $\text{Spec}_p(T)$). *In the setting of definitions 1.2 and 1.3 denote, for every $q > 0$ and $\alpha \in \mathbb{T}$*

$$V_{\alpha}^q := \{X \in L_{\mathbb{P}}^q : TX = \alpha X\}. \tag{1.2}$$

and let $V_{\alpha} := \cup_{q>0} V_{\alpha}^q$. Then the following statements are equivalent

1. $\alpha \in \text{Spec}_p(T)$.
2. $V_{\alpha} \neq \{0\}$.
3. $V_{\alpha} \cap L_{\mathbb{P}}^{\infty} \neq \{0\}$.

In particular, given $q > 0$, $\text{Spec}_p(T)$ is the set of eigenvalues of the Koopman operator $T : L_{\mathbb{P}}^q \rightarrow L_{\mathbb{P}}^q$.

Proof: Only 2. \Rightarrow 3. requires a proof.

Indeed, note that if $0 \neq Y \in V_{\alpha}$ is given, then from $|TY| = |\alpha Y| = |Y|$ \mathbb{P} -a.s. it follows that for all $M \geq 0$

$$TI_{|Y| \leq M} = I_{|Y| \leq M}T = I_{|TY| \leq M} = I_{|Y| \leq M}$$

\mathbb{P} -a.s. and therefore, choosing M such that $0 \neq I_{[|Y| \leq M]}$,

$$T(YI_{[|Y| \leq M]}) = (TY)(TI_{[|Y| \leq M]}) = \alpha YI_{[|Y| \leq M]}.$$

Thus $X := YI_{[|Y| \leq M]} \in V_\alpha$. Since clearly $X \in L^\infty_\mathbb{P} \setminus \{0\}$ this gives the desired conclusion. \square

Throughout this monograph, we will be mainly concerned with *ergodic transformations*. Ergodic transformations enjoy some special properties and, in some sense, they are the building blocks of any measure preserving transformation (see for instance Theorem 6 in [42]). The definition is the following.

Definition 1.4 (Ergodic Transformation). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A transformation $T : \Omega \rightarrow \Omega$ is called **ergodic** if it is measure preserving and every T -invariant set in \mathcal{F} is “trivial”. This is, if for every $A \in \mathcal{F}$: $T^{-1}A = A$ implies that $\mathbb{P}(A) \in \{0, 1\}$. Equivalently, if for all $A \in \mathcal{F}$: $TI_A = I_A$ implies that $EI_A \in \{0, 1\}$.*

Ergodicity is a spectral property: among its well-known characterizations the following one will be of special interest for us.

Lemma 1 (Ergodicity as a Spectral Property). *A measure-preserving transformation T on a probability space is ergodic if and only if 1 is a simple eigenvalue of T : if X satisfies $TX = X$ then X is (a.e. equal to a fixed) constant.*

Proof: See for instance [27], Proposition 2.14. \square

Our attention along this work will be mainly focused on Koopman operators associated to ergodic transformations on a probability space. To give a first consequence of the ergodic hypothesis note the following: according to the first line in the proof of Proposition 1.1, if Y is an eigenvector of T then $|Y|$ is T -invariant, and therefore constant if T is ergodic (Lemma 1). This gives the following result.

Proposition 1.2 (Circularity of Eigenfunctions). *Assume that T is ergodic and $\alpha \in \mathbb{T}$ is given: if Y satisfies $TY = \alpha Y$ then $|Y|$ is constant.*

Even more is true: the following proposition implies that, when T is ergodic, the eigenfunctions of T are unique up to multiplication by a scalar. Note also that in this case $\text{Spec}_p(T)$ is more than just a subset of \mathbb{T} .

Proposition 1.3 (Group Structure of $\text{Spec}_p(T)$). *With the notation in Proposition 1.1, and assuming T is ergodic, $\text{Spec}_p(T)$ is a subgroup of \mathbb{T} , and every element in $\text{Spec}_p(T)$ is a simple eigenvalue of T .*

Proof: The proposition consists of two statements, which we proceed to prove now.

$\text{Spec}_p(T)$ is a group. Since clearly $1 \in \text{Spec}_p(T)$ (consider any constant function X), it suffices to see that if $\alpha_1 \in \text{Spec}_p(T)$ and $\alpha_2 \in \text{Spec}_p(T)$, then $\alpha_1 \overline{\alpha_2} \in \text{Spec}_p(T)$.

Let us prove it: given $\alpha_1, \alpha_2 \in \text{Spec}_p(T)$ and nonzero functions $X_1 \in V_{\alpha_1}$ and $X_2 \in V_{\alpha_2}$, note that, since $|X_1|$ and $|X_2|$ are constant non-zero functions (Proposition 1.2), $X_1 X_2$ is (also) nonzero, and that

$$T(X_1 \overline{X_2}) = TX_1 \overline{TX_2} = \alpha_1 \overline{\alpha_2} X_1 X_2.$$

In particular $\alpha_1 \overline{\alpha_2} \in \text{Spec}_p(T)$, as claimed.

The eigenvalues are simple. If X, Y are (non-zero) eigenfunctions associated to $\alpha \in \text{Spec}_p(T) \subset \mathbb{T}$, the argument just given shows that $X\overline{Y}$ is an eigenfunction of T associated to 1. Since T is ergodic, there exists a constant $c \in \mathbb{C}$ such that $X\overline{Y} = c$, \mathbb{P} -a.s. It follows (multiply by Y) that $|Y|^2 X = cY$ and therefore, since $0 < |Y|$ is constant, there exists a constant $\beta (= c/\|Y\|_{\mathbb{P},\infty}^2)$ such that $X = \beta Y$. \square

1.2 Separability and Cardinality of the Point Spectrum

Let us recall now the following well known definition:

Definition 1.5 (Countably generated sigma-algebras). *A sigma algebra \mathcal{F} is countably generated if there exists a countable family of sets $\mathbb{A} = \{A_k\}_{k \in \mathbb{Z}} \subset \mathcal{F}$ such that $\sigma(\mathbb{A}) = \mathcal{F}$.*

This is the case if, for instance, \mathcal{F} is the Borel sigma algebra of a separable metric space (S, d) , or if \mathcal{F} is the sigma algebra generated by a countable family of random elements in a separable space. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and \mathcal{F} is countably generated, $L_{\mathbb{P}}^p$ is separable for every $p \in [1, \infty)$ (see [11], Theorem 19.2).

Now, it is a standard exercise to prove that the separability of $L_{\mathbb{P}}^2$ (or more generally, of any Hilbert space) is equivalent to the existence of a countable orthonormal basis of $L_{\mathbb{P}}^2$: a set $\{Y_k\}_{k \in \mathbb{Z}} \subset L_{\mathbb{P}}^2$ of mutually orthogonal elements whose linear span is dense in $L_{\mathbb{P}}^2$. In particular, if \mathcal{F} is countably generated, $L_{\mathbb{P}}^2$ admits at most countably many mutually orthogonal one-dimensional subspaces: for any family $\{Y_j\}_{j \in J} \subset L_{\mathbb{P}}^2$ of mutually orthogonal elements with $E[|Y_j|^2] = 1$, the balls centered at Y_j with radius 1 are mutually disjoint, which restricts the cardinality of J to a countable one if $L_{\mathbb{P}}^2$ is separable.

Recall the notation introduced in Proposition 1.1 and note that, since T is measure preserving, the spaces V_{α} are mutually orthogonal: given $\alpha_1 \in V_{\alpha_1}$ and $\alpha_2 \in V_{\alpha_2}$,

$$E[Y_1 \overline{Y_2}] = E[T[Y_1 \overline{Y_2}]] = \alpha_1 \overline{\alpha_2} E[Y_1 \overline{Y_2}]$$

which implies that either $\alpha_1 = \alpha_2$ or $E[Y_1 \overline{Y_2}] = 0$.

From these observations the following follows at once.

Proposition 1.4 (Cardinality of $\text{Spec}_p(T)$). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If \mathcal{F} is countably generated then for every measure-preserving transformation $T : \Omega \rightarrow \Omega$, $\text{Spec}_p(T)$ is countable. In particular*

$$\lambda(\{\theta \in [0, 2\pi) : e^{i\theta} \in \text{Spec}_p(T)\}) = 0. \quad (1.3)$$

2 Random Elements in L^2

In this section, we will introduce the results from Harmonic Analysis that will be used along the monograph. In particular, we will show how to use Carleson theorem (Theorem 2.1) to show that a random function $\omega \mapsto f_{\omega}$ in L_{λ}^2 (see Definition 2.3) defined on

$(\Omega, \mathcal{F}, \mathbb{P})$ induces a random function $([0, 2\pi), \mathcal{B}, \lambda) \rightarrow L^2_{\mathbb{P}}$ if the sigma algebra \mathcal{F} is countably generated (Theorem 2.2), a construction that will be important when justifying that the approximating martingales present along the proofs of the results in Chapter 4 are well defined.

On doing so, we will stop to discuss the measurability of a function defined by limits in a complete and separable metric space (Section 2.1). We will also introduce the notion of discrete Fourier Transforms (Definition 2.6) of a stochastic process, a generalization of the notion of partial sums that is at the heart of the results presented in this work.

2.1 Functions Defined by Limits

In this section, we will discuss the issue of the measurability for a map given by pointwise convergence of random functions in a metric space, and we will define the notion of “limit function” for an a.s convergent sequence of random elements in a complete and separable metric space in an unambiguous way. The results and definitions introduced here will be used, several times in an implicit way, along the discussions involving functions defined by (a.e.) convergent sequences.

We begin our discussion introducing the following technical notion.

Definition 2.1 (Distance to a set, ϵ -Neighborhood). *If (S, d) is a metric space with metric d , then for any given $x \in S$ and $A \subset S$ we define the **distance from x to A** by*

$$d(x, A) := \inf_{a \in A} d(x, a), \quad (1.4)$$

and we define the ϵ -neighborhood of A , A^ϵ , as the (open) set

$$A^\epsilon := \{x \in S : d(x, A) < \epsilon\}. \quad (1.5)$$

Assume that (S, d) is a (nonempty) metric space. In addition assume that (S, d) is complete and separable¹, let \mathcal{S} be the Borel sigma-algebra of S , and fix $s \in S$. If $(f_n)_{n \in \mathbb{N}}$ is a sequence of \mathcal{F}/\mathcal{S} measurable functions, define $C_{(f_n)_n}$ as the (measurable) set where $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Thus

$$C_{(f_n)_n} := \bigcap_{m \in \mathbb{N}^*} \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} [d(f_n, f_k) < 1/m], \quad (1.6)$$

and define the “limit function” $\lim_n f_n$ by

$$\omega \mapsto \begin{cases} \lim_n f_n(\omega) & , \text{ if } \omega \in C_{(f_n)_n} \\ s & , \text{ if } \omega \notin C_{(f_n)_n} \end{cases} \quad (1.7)$$

¹The assumption of completeness is made to guarantee that the set of points where a given sequence of functions converges is measurable. The assumption of separability is made to guarantee that $\mathcal{S} \otimes \mathcal{S}$ is the Borel sigma-algebra of $S \times S$ (see Appendix M10 in [10]), so that the distance function $d : S \times S \rightarrow [0, \infty)$, which is continuous with respect to the product topology, is $\mathcal{S} \otimes \mathcal{S}$ -measurable, and for any two given \mathcal{F}/\mathcal{S} measurable functions f, g , the function $\omega \mapsto d(f(\omega), g(\omega))$ is \mathcal{F} -measurable.

Now remember the well known definition of the \liminf of a family of (measurable) sets $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$:

$$\liminf_n A_n := \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} A_k,$$

and note that that for every closed set $F \subset S$

$$(\lim_n f_n)^{-1}(F) = \begin{cases} C_{(f_n)_n} \cap \bigcap_{m \in \mathbb{N}^*} \liminf_n [f_n \in F^{1/m}] & , \text{ if } s \notin F \\ (\Omega \setminus C_{(f_n)_n}) \cup (C_{(f_n)_n} \cap \bigcap_{m \in \mathbb{N}^*} \liminf_n [f_n \in F^{1/m}]) & , \text{ if } s \in F. \end{cases} \quad (1.8)$$

The measurability of these sets, together with the $\pi - \lambda$ theorem (applied to the set of elements $A \in \mathcal{S}$ such that $(\lim_n f_n)^{-1}(A) \in \mathcal{F}$) give at once the following result.

Proposition 2.1 (Measurability of Limit Functions). *Let (S, d) be a complete and separable (nonempty) metric space with Borel sigma algebra \mathcal{S} . Given any sequence $(f_n)_{n \in \mathbb{N}}$ of \mathcal{F}/\mathcal{S} measurable functions defined on some measurable space (Ω, \mathcal{F}) , the function $\lim_n f_n$ defined by (1.7) is \mathcal{F}/\mathcal{S} -measurable.*

Finally note that if \mathbb{P} is a probability measure on (Ω, \mathcal{F}) and $\mathbb{P}(\Omega \setminus C_{(f_n)_n}) = 0$, then the \mathbb{P} -equivalence class of $\lim_n f_n$ is independent of the choice of s .

Let us formalize this in the following definition

Definition 2.2 (Functions Defined by Limits). *In the context of Proposition 2.1, assume that \mathbb{P} is a probability measure on (Ω, \mathcal{F}) , and that $\mathbb{P}(\Omega \setminus C_{(f_n)_n}) = 0$. We define **the** limit function (also denoted by) $\lim_n f_n$ as the \mathbb{P} -equivalence class of functions represented by $\lim_n f_n$.*

Let us finish this section by reminding the formal notion of a random element in a metric space.

Definition 2.3 (Random Elements and their Law). *If (S, d) is a metric space with Borel sigma algebra \mathcal{S} , a **random element of S** is an \mathcal{F}/\mathcal{S} measurable function $V : \Omega \rightarrow S$ from some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to (S, \mathcal{S}) . If V is a random element on S , the **law of V** is the probability measure $\mathbb{P}V^{-1}$ on \mathcal{S} defined by*

$$\mathbb{P}V^{-1}(A) = \mathbb{P}[V \in A]$$

for all $A \in \mathcal{S}$.

During the rest of this section, we will focus our attention on random elements in L_λ^2 . This is, \mathcal{F}/\mathcal{S} measurable functions $V : \Omega \rightarrow L_\lambda^2$ from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the space (S, d) of square-integrable functions in $([0, 2\pi], \mathcal{B}, \lambda)$ with the L_λ^2 norm.

2.2 The Fourier Transform of an Integrable Function

Let us begin this section by reminding the notion of the *Fourier transform* of a function $f \in L_\lambda^1$, which is the building block for the representation by Fourier series of elements in L_λ^2 (or, under an appropriate notion of convergence, of elements in L_λ^1).

Definition 2.4 (Fourier Transform). *Given $f \in L^1_\lambda$, $\hat{f} : \mathbb{R} \rightarrow \mathbb{C}$ will denote the **Fourier transform of f** , which is defined by*

$$\hat{f}(x) = \int_0^{2\pi} f(\theta) e^{-ix\theta} d\lambda(\theta). \quad (1.9)$$

Our first goal is to describe in which sense the Fourier transform of a function allows us to represent it in a convenient way. The first step towards this goal is to define the *Fourier partial sums* of a function in L^1_λ .

Definition 2.5 (Fourier Partial Sums). *For a given $n \in \mathbb{N}^*$, the n -th **Fourier partial sum of a function $f \in L^1_\lambda$** at a frequency $\theta \in [0, 2\pi)$ is defined by*

$$S_{f,n}(\theta) := \sum_{k=1-n}^{n-1} \hat{f}(k) e^{ik\theta}. \quad (1.10)$$

In 1966 Lennart Carleson ([15]) proved the following celebrated result, establishing that the Fourier series representation of a function in L^2_λ is convergent almost surely.²

Theorem 2.1 (Carleson). *Let $f \in L^2_\lambda$ and let $S_{f,n}(\theta)$ be defined by (1.10), then*

$$f(\theta) = \lim_n S_{f,n}(\theta)$$

in the sense of Definition 2.2. This is: there exists a set I_f with $\lambda(I_f) = 1$ such that for every $\theta \in I_f$, $\lim_n S_{f,n}(\theta) = f(\theta)$.

Now, given $f \in L^2_\lambda$, Parseval's theorem ([32], Proposition 3.1.16, (3)) establishes that

$$\int_0^{2\pi} |f(\theta)|^2 d\lambda(\theta) = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2$$

and, reciprocally, Plancherel's theorem ([32], Proposition 3.1.16, (2) and (4)) establishes that for any $(c_k)_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$, the map (λ -a.e) given by

$$\theta \mapsto \sum_{k \in \mathbb{Z}} c_k e^{ik\theta} \quad (1.11)$$

defines a (unique) element $f \in L^2_\lambda$, with Fourier coefficients $\hat{f}(k) = c_k$. These observations can be summarized in the following proposition.

Proposition 2.2 (Representation of L^2_λ). *The correspondence $L^2_\lambda \rightarrow l^2(\mathbb{Z})$ given by $f \mapsto (\hat{f}(k))_{k \in \mathbb{Z}}$ is (well defined and) bijective.*

Remark 2.1. Note that, by Parseval's Theorem, the correspondence given in Proposition 2.2 is an isometry of metric spaces.³

²This result is also true for functions in L^p_λ with $p > 1$. See for instance [36].

³This is actually the content of [32], Proposition 3.1.16, (4).

2.3 A Duality Theorem

In virtue of Proposition 2.2 and Remark 2.1 we can think of functions in L_λ^2 just as elements in $l^2(\mathbb{Z})$. In particular, a random function in L_λ^2 can be thought of as a measurable map

$$\mathbf{Y} : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow l^2(\mathbb{Z})$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Concretely, given a random function $\omega \mapsto f_\omega$ of L_λ^2 , take

$$\mathbf{Y}(\omega) := (\hat{f}_\omega(k))_{k \in \mathbb{Z}},$$

where \hat{f}_ω is the Fourier Transform of f_ω (Definition 2.4).

Reciprocally, since $l^2(\mathbb{Z})$ is separable (see the discussion in Section 1.2), a random element in L_λ^2 is specified by any sequence $(Y_k)_{k \in \mathbb{Z}}$ of random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$, provided that

$$\sum_{k \in \mathbb{Z}} |Y_k|^2 < \infty \quad \mathbb{P}\text{-a.s.} \quad (1.12)$$

where $\mathbf{Y}(\omega) := 0$ if ω does not belong to the set where (1.12) converges.

More can be said in this case: since for \mathbb{P} -almost every ω , the series

$$\sum_{k \in \mathbb{Z}} Y_k(\omega) e^{ik\theta}$$

is λ -a.e convergent, the $\mathcal{B} \otimes \mathcal{F}$ -set

$$A := \{(\theta, \omega) \in [0, 2\pi) \times \Omega : \sum_{k \in \mathbb{Z}} Y_k(\omega) e^{ik\theta} \text{ is convergent}\}$$

satisfies $\lambda \otimes \mathbb{P}(A) = 1$, and an application of Fubini's theorem shows that there exists a set $I_Y \subset [0, 2\pi)$ with $\lambda(I_Y) = 1$ satisfying following property: for every $\theta \in I_Y$ there exists Ω_θ with $\mathbb{P}(\Omega_\theta) = 1$ such that the series

$$\sum_{k \in \mathbb{Z}} Y_k e^{ik\theta} \quad (1.13)$$

converges for all $\omega \in \Omega_\theta$.

If we assume in addition that, for a given $\theta \in I_Y$ (or in a set $I'_Y \subset I_Y$ with $\lambda(I'_Y) = 1$)

$$E \left[\sup_{n \in \mathbb{N}} \left| \sum_{k=1-n}^{n-1} Y_k(\omega) e^{ik\theta} \right|^2 \right] < \infty \quad (1.14)$$

then, by Lebesgue's dominated convergence theorem, the function given by

$$\omega \mapsto \sum_{k \in \mathbb{Z}} Y_k(\omega) e^{ik\theta}$$

belongs to $L_\mathbb{P}^2$. In particular we have the following result.

Theorem 2.2 (Duality of Random Elements in L^2). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Assume that \mathcal{F} is countably generated (Definition 1.5), and let $\mathbf{Y} : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow l^2(\mathbb{Z})$ be a random element of $l^2(\mathbb{Z})$. If (1.14) holds for λ -a.e θ , then the map $(\lambda$ -a.e) defined by*

$$\theta \mapsto \sum_{k \in \mathbb{Z}} Y_k e^{ik\theta} \quad (1.15)$$

(where the series is defined in the \mathbb{P} -a.s sense) is a random element $([0, 2\pi), \mathcal{B}, \lambda) \rightarrow L^2_{\mathbb{P}}$ of $L^2_{\mathbb{P}}$.

If $Y : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow L^2_{\lambda}$ is a random element of L^2_{λ} and we denote by \hat{f} the Fourier transform of f (Definition 2.4), the same statement holds by taking $\mathbf{Y}(\omega) = (Y_k(\omega))_{k \in \mathbb{Z}} := (\widehat{Y(\omega)}(k))_{k \in \mathbb{Z}}$.

Proof: First: since \mathcal{F} is countably generated, $L^2_{\mathbb{P}}$ is (complete and) separable (see the comments following Definition 1.5).

Let now $\mathcal{L}^2_{\mathbb{P}}$ denote the Borel sigma-algebra of $L^2_{\mathbb{P}}$. Only the $\mathcal{B}/\mathcal{L}^2_{\mathbb{P}}$ measurability of (1.15) is left to prove, which will follow if we can prove that the convergence of (1.13) in the \mathbb{P} -a.s sense (which is guaranteed for λ -a.e θ) together with (1.14) implies the convergence of (1.15) in the $L^2_{\mathbb{P}}$ -sense for λ -a.e θ .⁴

To see this we can argue as follows: by the a.s convergence of (1.13)

$$\lim_N \left| \sum_{|k| \leq N} Y_k e^{ik\theta} - \sum_{k \in \mathbb{Z}} Y_k e^{ik\theta} \right| = 0$$

\mathbb{P} -a.s for λ -a.e θ , and since for every $N \in \mathbb{N}$ (and every such θ)

$$\left| \sum_{|k| \leq N} Y_k e^{ik\theta} - \sum_{k \in \mathbb{Z}} Y_k e^{ik\theta} \right|^2 \leq 2 \sup_{n \in \mathbb{N}} \left| \sum_{|k| < n} Y_k e^{ik\theta} \right|^2,$$

the dominated convergence theorem, together with (1.14), imply that

$$\lim_N E \left| \sum_{|k| \leq N} Y_k e^{ik\theta} - \sum_{k \in \mathbb{Z}} Y_k e^{ik\theta} \right|^2 = 0$$

for λ -a.e θ , as desired.

The last statement follows at once from the previous one combined with Proposition 2.2. \square .

⁴More precisely, note that for given $N \in \mathbb{N}$, the map $f_N : [0, 2\pi) \rightarrow L^2_{\mathbb{P}}$ given by

$$f_N(\theta) := \sum_{|k| \leq N} Y_k e^{ik\theta}$$

is $\mathcal{B}/\mathcal{L}^2_{\mathbb{P}}$ measurable (it is indeed continuous), and that if (1.15) makes sense as a limit in $L^2_{\mathbb{P}}$ for λ -a.e θ , then it is indeed the same as the map $f := \lim_N f_N$ (in the sense of Definition 2.2).

2.4 Duality via Decay of Second Moments

In this section we will give a sufficient condition (see (1.17) below) to guarantee the fulfillment of (1.14), and therefore the validity of the conclusion of Theorem 2.2. We introduce also the notion of the $(n\text{--th})$ *discrete Fourier transform* of a stationary process, whose normalized asymptotic behavior is the main topic of this work.

A Maximal Inequality, the Discrete Fourier Transforms

The following result is another classical tool in Harmonic Analysis (we give here a particular version sufficient for our purposes).

Theorem 2.3 (A Maximal Inequality). *There exists a constant C with the following property: for all $f \in L^2_\lambda$*

$$\int_0^{2\pi} \sup_{n \in \mathbb{N}^*} |S_{f,n}(\theta)|^2 d\lambda(\theta) \leq C \sum_{k \geq 0} |\hat{f}(k)|^2, \quad (1.16)$$

where $S_{f,n}(\theta)$ is the $n\text{--th}$ Fourier partial sum of f at θ (see (1.10)) and \hat{f} denotes the Fourier transform of f (Definition 2.4).

Proof: See [33]. □

From now on, we will refer to the inequality (1.16) as *Hunt and Young's maximal inequality*.

Now consider the following condition on a stochastic process $(Y_k)_{k \in \mathbb{Z}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\sum_{k \in \mathbb{Z}} \|Y_k\|_{\mathbb{P}, 2}^2 < \infty. \quad (1.17)$$

Note that, under this condition, (1.12) is satisfied (the function $\sum_{k \in \mathbb{Z}} |Y_k|^2$ is actually integrable by the monotone convergence theorem), and $(Y_k)_{k \in \mathbb{Z}}$ is therefore a random element of $l^2(\mathbb{Z})$. Even more, by Theorem 2.3, there exists a constant C such that

$$\int_0^{2\pi} \sup_n \left| \sum_{k=1-n}^{n-1} Y_k e^{ik\theta} \right|^2 d\lambda(\theta) \leq C \sum_{k \in \mathbb{Z}} |Y_k|^2 \quad \mathbb{P}\text{-a.s.} \quad (1.18)$$

More precisely, (1.18) holds on the set of \mathbb{P} –measure one

$$[\sum_{k \in \mathbb{Z}} |Y_k|^2 < \infty].$$

Integrating with respect to \mathbb{P} , and using Fubini's theorem we get that, under (1.14),

$$\int_0^{2\pi} E[\sup_n \left| \sum_{k=1-n}^{n-1} Y_k e^{ik\theta} \right|^2] d\lambda(\theta) \leq C \sum_{k \in \mathbb{Z}} E[|Y_k|^2] < \infty.$$

In particular

$$E[\sup_n |\sum_{k=1-n}^{n-1} Y_k e^{ik\theta}|^2] < \infty \quad (1.19)$$

for λ -a.e θ . This, combined with Theorem 2.2 gives the following result.

Proposition 2.3 (A Criterion for Duality). *If a stochastic process $(Y_k)_{k \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfies (1.17) and \mathcal{F} is countably generated, the function*

$$\theta \mapsto \sum_{k \in \mathbb{Z}} Y_k e^{ik\theta} \quad (1.20)$$

defines a random element $([0, 2\pi), \mathcal{B}, \lambda) \rightarrow L_{\mathbb{P}}^2$, and there exists $I' \subset [0, 2\pi)$ with $\lambda(I') = 1$ such that for every $\theta \in I'$, (1.19) is verified and (1.20) converges \mathbb{P} -a.s .

Remark 2.2. The assumption on \mathcal{F} (being countably generated) is made only to prove the $\mathcal{B}/\mathcal{L}_{\mathbb{P}}^2$ measurability of the map (1.20) (see the proof of Theorem 2.2): the existence of the set I' holds regardless of the nature of \mathcal{F} .⁵

Before continuing with our discussion, let us stop here to introduce the notion of *discrete Fourier Transforms* of a stochastic process.

Definition 2.6 (Discrete Fourier Transforms). *Given a stochastic process $(Y_k)_k$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $n \in \mathbb{N}^*$, we will define the **n -th discrete Fourier transform of $(Y_k)_k$ at the frequency $\theta \in [0, 2\pi)$, $S_n(\theta, \cdot) : \Omega \rightarrow \mathbb{C}$, by***

$$S_n(\theta, \omega) := \sum_{k=0}^{n-1} Y_k(\omega) e^{ik\theta}. \quad (1.21)$$

If $(Y_k)_k$ is not clear from the context, we will use the notation $S_n((Y_k)_k, \theta, \omega)$ for $S_n(\theta, \omega)$. If θ is fixed, we will denote by $S_n(\theta)$ the random variable $S_n(\theta, \cdot)$. If $\theta = 0$, we denote by S_n the random variable $S_n(0, \cdot)$.

Remark 2.3 (A note on the definition of $S_n(\theta)$). As the reader can see, if $(Y_k)_{k \in \mathbb{Z}}$ is a process indexed by \mathbb{Z} , we are not including the elements Y_k for $k < 0$ in our notion of discrete Fourier Transforms.

A plausible alternative may be to sum over the set of indexes $\{1 - n, \dots, n - 1\}$ but, while (1.21) may certainly be an example of a “bad definition” in the framework of a more general theory, we stick to it here mainly due to the facts that, first, all of our forthcoming discussions will be made under the additional hypothesis that $(X_k)_k$ is strictly stationary

⁵Note that for every $N \in \mathbb{N}$ the map

$$(\theta, \omega) \mapsto \sum_{|k| \leq N} Y_k(\omega) e^{ik\theta}$$

is $\mathcal{B} \otimes \mathcal{F}$ -measurable. So is $(\theta, \omega) \mapsto f_n(\theta, \omega) := \max_{0 \leq k \leq n} |\sum_{j=1-k}^{k-1} Y_j e^{ik\theta}|^2$, and therefore so is $(\theta, \omega) \mapsto \lim_n f_n(\theta, \omega)$ (in the sense of (1.7)).

It is then clear that the last map is $\mathcal{B} \otimes \mathcal{F}$ -measurable and, under (1.17), it coincides $\lambda \times \mathbb{P}$ -a.s with $\sup_{n \geq 0} |\sum_{j=1-n}^{n-1} Y_j(\omega) e^{ik\theta}|^2$ by (1.18). The argument for the existence of I' goes through just as explained.

(Definition 4.1), which allows us to generate the process $(X_k)_{k \in \mathbb{Z}}$ knowing only the initial function X_0 and, second, our results will be concerned with the asymptotics related to “ $S_n(\theta) - E_0 S_n(\theta)$ ” (see Section 4 for the corresponding notation), a normalization that would annihilate the summands with negative index in the “extended” definition of the discrete Fourier Transforms.

Finally, our theory is concerned with the asymptotics of processes “with initial time”, an assumption implicit along the tradition of the study of central limit theorems, and even more important here given the heuristics of the notion of convergence that we will deal with (see Section 4.2).

There are other practical reasons to keep this definition (for instance: we would encounter problems passing from the “randomly centered” to the “non-centered” case along the discussion in Section 16 if we adopted the extended definition of $S_n(\theta)$) but after all this choice is, to a certain extent, just a matter of taste, and it is a good exercise for the reader to verify which of the proofs concerning $S_n(\theta)$ can be carried through with the suggested, more symmetric definition of the discrete Fourier transforms.

3 Dunford-Schwartz Operators and the Ergodic Theorem

In this section we present the ergodic theorem for positive Dunford-Schwartz operators and its consequent ergodic theorem for discrete Fourier transforms, a result that has interest in itself and justifies the investigation of the validity of the central limit theorem for the normalized averages of the discrete Fourier transforms of a stationary process. We also make a digression towards weak L^p -spaces, whose weak norms provide a framework that will be of use along the proofs of forthcoming results.

3.1 The Ergodic Theorem for Positive Dunford-Schwartz Operators

To begin with, let us recall the definition of a *Dunford-Schwartz operator*.

Definition 3.1 (Dunford-Schwartz Operators). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A **Dunford-Schwartz operator** $T : L_\mu^1 \rightarrow L_\mu^1$ is a linear operator with the following property: for every $p \geq 1$ and every $X \in L_\mu^p \cap L_\mu^1$*

$$\|TX\|_p \leq \|X\|_p . \quad (1.22)$$

Remark 3.1. It is possible to see ([26], Theorem 8.23) that if (1.22) holds for $p = 1$ and $p = \infty$ then T is Dunford-Schwartz. It is also clear that when $\mu(\Omega) < \infty$ (for instance if μ is a probability measure), this definition is equivalent to the condition that T is a contraction in L_μ^p for every $p \geq 1$ (use the well known continuous embedding $L_\mu^p \subset L_\mu^1$, valid when $\mu(\Omega)$ is finite).

We will also need to make use of the notion of *positive operator*.

Definition 3.2 (Positive Operators). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $T : L_\mu^1 \rightarrow L_\mu^1$ be a bounded linear operator. T is called **positive** if for any $X \in L_\mu^1$, $T|X|$ is non-negative.*

The following theorem arises from a combination of Theorems 8.24, 11.4 and 11.6 in [26], together with the fact that the operator T involved in the hypotheses is continuous in the respective L^p space.

Theorem 3.1 (The Mean and Pointwise Ergodic Theorem for Positive Dunford-Schwartz Operators). *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space ($\mu(\Omega) < \infty$) and let $T : L_\mu^1 \rightarrow L_\mu^1$ be a positive Dunford-Schwartz operator (Definitions 3.1 and 3.2). Then for every $p \geq 1$ and every $X \in L_\mu^p$ there exists $P_T X$ with the following properties*

1. $P_T X$ is T -invariant: $TP_T X = P_T X$.
2. The Cesaro-averages $(X + \cdots + T^{n-1}X)/n$ converge to $P_T X$ μ -a.s and in L_μ^p :

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} T^k X = P_T X \quad \mu\text{-a.s and in } L_\mu^p. \quad (1.23)$$

Proof: Denote by $A_n X$ ($n \geq 1$) the corresponding averages in the conclusion of Theorem 3.1. This is

$$A_n X := \frac{1}{n} \sum_{k=0}^{n-1} T^k X. \quad (1.24)$$

For the existence of the limit in statement 2. see to the proofs of Theorems 8.24, 11.4 and 11.6 in [26]. Denote this limit by $P_T X$.

To see that the limit satisfies 1. note that, since $A_n X$ converges in L_μ^p

$$TP_T X = \lim_n TA_n X$$

(here “lim” denotes limit in L_μ^p) and that, since $T^n X/n \rightarrow 0$ as $n \rightarrow \infty$ μ -a.s ($T^n X/n = (n+1)A_{n+1}/n - A_n$):

$$\lim_n (A_n X - TA_n X) = \lim_n \frac{1}{n} (X - T^n X) = 0, \quad \mu\text{-a.s.} \quad \square$$

Remark 3.2 (P_T as a projection, a case of orthogonality.). Given a Banach space B with norm $\|\cdot\|_B$, a **projection on B** is a continuous linear operator $P : B \rightarrow B$ with the property that $P^2 = P$. If P is a projection and $V_P := PB$, we say that P *projects B onto V_P* .

Notice that Theorem 3.1 can be stated in the following way: *let T be a nonnegative Dunford-Schwartz operator and, for $p \geq 1$, let $V_{T,p} \subset L_\mu^p$ be the (closed) subspace of T -invariant functions ($Y \in V_{T,p}$ if and only if $TY = Y$, \mathbb{P} -a.s.). Then the function $P_T : L_\mu^p \rightarrow V_{T,p}$ given by*

$$P_T Y = \lim_n A_n Y$$

is well defined both in the \mathbb{P} -a.s. and $L_{\mathbb{P}}^p$ -senses.

It is easy to see that, indeed, $P_T L_{\mathbb{P}}^p = V_{T,p}$, and P_T is clearly linear. Since P_T is a contraction in $L_{\mathbb{P}}^p$ ($\|\lim_n A_n Y\|_p = \lim_n \|A_n Y\|_p \leq \|Y\|_p$), and since $P_T^2 = P_T$ (by the T -invariance of $P_T X$ for every X), P_T is a projection.

Assume now that T preserves the mean: for every $p \geq 1$ and every $Y \in L_{\mathbb{P}}^p$

$$E[TY] = E[Y], \quad (1.25)$$

and assume in addition that, either T is multiplicative

$$T[XY] = TXY \quad (1.26)$$

(this is the case for instance when T is a Koopman operator) or that

$$T[XTY] = TXTY \quad (1.27)$$

(for instance if T is a conditional expectation), whenever the expressions involved make sense. Then we can show that, actually, P_T is **orthogonal**. This is, that if $p \in [1, \infty)$ is given, then for every $X \in L_{\mathbb{P}}^p$ and $Y \in L_{\mathbb{P}}^{p/(p-1)}$ ($Y \in L_{\mathbb{P}}^{\infty}$ if $p = 1$), $E[(X - P_T X)P_T Y] = 0$.

To do so we notice the following: first, since $P_T Z = \lim_n A_n Z$ in the $L_{\mathbb{P}}^1$ sense,

$$E[P_T Z] = E[\lim_n A_n Z] = \lim_n E[A_n Z] = E[Z] \quad (1.28)$$

for every $Z \in L_{\mathbb{P}}^1$. Then, since $P_T Y$ is T -invariant,

$$T^n(X P_T Y) = (T^n X) P_T Y \quad (1.29)$$

where n is any natural number⁶, and therefore $P_T X P_T Y = P_T(X P_T Y)$.

All together, this gives that for every $p \geq 1$, $X \in L_{\mathbb{P}}^p$ and $Y \in L_{\mathbb{P}}^{p/(p-1)}$:

$$\begin{aligned} E[(X - P_T X)(\overline{P_T Y})] &= E[X P_T \overline{Y}] - E[(P_T X)(P_T \overline{Y})] = E[X P_T \overline{Y}] - E[P_T(X P_T \overline{Y})] = \\ &= E[X P_T \overline{Y}] - E[X P_T \overline{Y}] = 0 \end{aligned}$$

as claimed.

3.2 The Ergodic Theorem for Discrete Fourier Transforms

The results provided in section 3.1 allow us to generalize the mean and pointwise ergodic theorems to the case of rotated partial sums (discrete Fourier transforms). In particular,

⁶This is obvious under (1.26), and to prove it under (1.27) we proceed by induction: the case $n = 0$ is trivial, an assuming that (1.29) holds for a value of n :

$$\begin{aligned} T^{n+1}(X P_T Y) &= T(T^n(X P_T Y)) = T((T^n X) P_T Y) = T((T^n X) T(P_T Y)) = (T^{n+1} X)(T P_T Y) = \\ &= (T^{n+1} X) P_T Y. \end{aligned}$$

this justifies an interpretation of the main results of this monograph as theorems about the “speed of convergence” for the normalized averages of the discrete Fourier transforms.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $T : \Omega \rightarrow \Omega$ be an invertible, bimeasurable measure-preserving transformation, and let $\theta \in [0, 2\pi)$ be given.

Consider the transformation $\tilde{T}_\theta : [0, 2\pi) \times \Omega \rightarrow [0, 2\pi) \times \Omega$ specified by

$$\tilde{T}_\theta(u, \omega) = ((u + \theta) \bmod(2\pi), T\omega). \quad (1.30)$$

Notice that \tilde{T}_θ is simply the product map between the rotation $u \mapsto (u + \theta) \bmod(2\pi)$ and T . This transformation is clearly measure preserving and invertible.

Definition 3.3 (Extension to the Product Space). *Let $p \geq 1$ and $Y \in L^p_{\mathbb{P}}$ be given, we will denote by \tilde{Y} the extension of Y to $[0, 2\pi) \times \Omega$ given by the following formula:*

$$\tilde{Y}(u, \omega) = e^{iu} Y(\omega).$$

It is clear that $\tilde{Y} \in L^p_{\lambda \times \mathbb{P}}$ and that the $L^p_{\lambda \times \mathbb{P}}$ norm of this extension is the same as the $L^p_{\mathbb{P}}$ norm of Y . Note also that if \tilde{T}_θ is given by (1.30), then for all $k \in \mathbb{Z}$,

$$\tilde{T}_\theta^k \tilde{Y} = e^{ik\theta} \widetilde{T^k Y} \quad (1.31)$$

Note that \tilde{T}_θ , seen as an operator in $L^p_{\lambda \times \mathbb{P}}$ for $p \in [1, +\infty)$ (namely $Z \in L^p_{\lambda \times \mathbb{P}} \mapsto Z \circ \tilde{T}_\theta$: the Koopman operator associated to \tilde{T}_θ), is a positive contraction for every p . It follows from Theorem 3.1 that there exists a \tilde{T}_θ -invariant function $\tilde{P}_\theta \tilde{Y}$ such that

$$\frac{1}{n} \sum_{k=0}^{n-1} \tilde{T}_\theta^k \tilde{Y}(u, \omega) = \frac{e^{iu}}{n} S_n((T^k Y)_{k \in \mathbb{N}}, \theta)(\omega) \rightarrow_n \tilde{P}_\theta \tilde{Y}(u, \omega)$$

$\lambda \times \mathbb{P}$ -a.s and in $L^p_{\lambda \times \mathbb{P}}$. Fixing u_0 such that $\sum_{k=0}^{n-1} \tilde{T}_\theta^k \tilde{Y}(u_0, \cdot)/n$ converges \mathbb{P} -a.s we see that, if

$$P_\theta Y(\omega) := e^{-iu_0} \tilde{P}_\theta \tilde{Y}(u_0, \omega)$$

then, necessarily

$$\tilde{P}_\theta \tilde{Y}(u, \omega) = e^{iu} P_\theta Y(\omega), \quad \lambda \times \mathbb{P}\text{-a.s.}$$

In particular, $\widetilde{P_\theta Y} = \tilde{P}_\theta \tilde{Y}$.

Finally note that, since $\widetilde{P_\theta Y}$ is \tilde{T}_θ -invariant

$$TP_\theta Y = e^{-iu} e^{-i\theta} \widetilde{\tilde{T}_\theta P_\theta Y} = e^{-iu} e^{-i\theta} \tilde{T}_\theta(\tilde{P}_\theta \tilde{Y}) = e^{-i\theta} e^{-iu} \tilde{P}_\theta \tilde{Y} = e^{-i\theta} P_\theta Y.$$

This proves the following result.

Theorem 3.2 (The Ergodic Theorem for Discrete Fourier Transforms). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $T : \Omega \rightarrow \Omega$ a measure-preserving transformation, $p \geq 1$, $Y \in L^p_{\mathbb{P}}$, and denote (also) by T the Koopman operator associated to T (Definition 1.2) and by $S_n(Y, \theta)$ the n -th discrete Fourier transform of the process $(T^k Y)_{k \in \mathbb{N}}$ (Definition 2.6). Then for every $\theta \in [0, 2\pi)$ there exists a function $P_\theta Y \in L^p_{\mathbb{P}}$ with the following properties*

1. $TP_\theta Y = e^{-i\theta} P_\theta Y$, \mathbb{P} -a.s.
2. $S_n(Y, \theta)/n \rightarrow_n P_\theta Y$, \mathbb{P} -a.s. and in $L_{\mathbb{P}}^p$.

If T is ergodic $|P_\theta Y|$ is constant and $P_\theta Y$ is unique up to a scalar multiple. This is: if $Y_1, Y_2 \in L_{\mathbb{P}}^p$, then there exists $c \in \mathbb{C}$ such that $P_\theta Y_1 = cP_\theta Y_2$.

Proof: By the preceding discussion, only the last statement requires a proof, but this follows at once from Proposition 1.2 and Colollary 1.3. \square

Remark 3.3 (P_θ as an orthogonal projection). It is easy to see that, in general,

$$E_{\mathbb{P}}[X\bar{Y}] = E_{\lambda \times \mathbb{P}}[\tilde{X}\bar{\tilde{Y}}].$$

By Remark 3.2, \tilde{P}_θ is an orthogonal projection onto the subspace $V_{\tilde{T}_\theta} \subset L_{\lambda \times \mathbb{P}}^p$ of functions that are invariant with respect to \tilde{T}_θ . It follows that for every $X \in L_{\mathbb{P}}^p$, $Y \in L_{\mathbb{P}}^{p/(p-1)}$

$$E_{\mathbb{P}}[(X - P_\theta X)\overline{P_\theta Y}] = E_{\lambda \times \mathbb{P}}[(\tilde{X} - \tilde{P}_\theta \tilde{X})\overline{\tilde{P}_\theta \tilde{Y}}] = 0.$$

This is: for fixed $p \geq 1$, P_θ is the orthogonal projection onto $V_{\theta,p}$, where

$$V_{\theta,p} := \{Y \in L_{\mathbb{P}}^p : TY = e^{-i\theta} Y\}.$$

In particular, taking $\theta = 0$, we get the classical statement of the mean and pointwise ergodic theorems for stationary sequences.

We remark also the following corollary.

Corollary 3.3. *With the notation of Theorem 3.2, if $e^{-i\theta} \notin \text{Spec}_p(T)$ (equivalently, if $e^{i\theta} \notin \text{Spec}_p(T)$), then*

$$\frac{1}{n} S_n(Y, \theta) \rightarrow 0 \quad \mathbb{P}\text{-a.s. and in } L_{\mathbb{P}}^p.$$

Proof: This is a trivial consequence of the definition of $\text{Spec}_p(T)$ and the statement 1. in Theorem 3.2. \square

What is the speed of convergence of the averages in Theorem 3.2 and Corollary 3.3? By considering the case $\theta = 0$ (the “classical” case) we see that this question does not admit an answer valid for any given $Y \in L_{\mathbb{P}}^p$ but, as shown by Peligrad and Wu in [41], the central limit theorem (CLT) holds for λ -a.e frequency $\theta \in [0, 2\pi)$ under the additional (standard) assumptions $Y \in L_{\mathbb{P}}^2$, $EY = 0$, and a certain regularity condition (see (1.58)) to be discussed later ⁷, this is Theorem 5.5 in this monograph. Peligrad and Wu show also that under these hypotheses the functional CLT (FCLT) also holds for *averaged* frequencies ([41], Theorem 2.1). This is Theorem 5.6 in page 43.

⁷Peligrad and Wu’s paper was preceded by Wu’s [47], in which the same result is proved under the additional assumption

$$\sum_{k>0} \frac{\|E[X_k | \mathcal{F}_0]\|_2}{k} < \infty,$$

where for every $k \in \mathbb{Z}$, $X_k = f(\xi_k)$ with $(\xi_k)_{k \in \mathbb{Z}}$ a stationary Markov Chain and $\mathcal{F}_0 = \sigma(\xi_k)_{k \leq 0}$ (see Section 12.2 for a discussion related to this setting).

Our main goal in this work is to adjust these CLTs to the quenched setting, and to give more precise information about the nature of the asymptotic distribution for a given frequency $\theta \in [0, 2\pi)$. We will also see that these “adjustments” are not simply an extension of the results just mentioned: not every centered process in $L^2_{\mathbb{P}}$ for which the CLT or the FCLT is valid admits quenched asymptotic distributions: the discrete Fourier transforms have to be (randomly) centered to remain orthogonal to the subspace of functions that are measurable with respect to the initial sigma-field (see Chapter 3 for the terminology). For a precise description of these results see Chapter 4.

3.3 Dunford-Schwartz Operators and the Weak L^p -spaces

The purpose of this section is to prove that, for Dunford-Schwartz operators, a variation of Hunt and Young’s maximal inequality (Theorem 2.3) is available if we refine the norm in L^p_{μ} to the *weak* (induced) norm in $L^{p,\infty}$ for $p > 1$. This result will be of importance to prove approximations involving the maxima of (normalized) partial sums via techniques akin to those involving Doob’s maximal inequality. We start this section by recalling the notion of the *weak L^p -spaces*, $L^{p,\infty}$.

Definition 3.4 (Weak L^p spaces). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and given a measurable function $Y : \Omega \rightarrow \mathbb{C}$ and $0 < p < \infty$, let $[Y]_{\mu,p}$ be given by*

$$[Y]_{\mu,p} := \sup_{\alpha > 0} (\alpha^p \mu(|Y| > \alpha)). \quad (1.32)$$

*We define the **weak L^p -space associated to μ** , $L^{p,\infty}_{\mu}$, as the topological space set-theoretically given by*

$$L^{p,\infty}_{\mu} := \{Y : [Y]_{\mu,p} < \infty\}, \quad (1.33)$$

and whose topology is induced by the quasi-norm $[\cdot]_{\mu,p}$. If $p = \infty$ we define the L^{∞} weak space by $L^{\infty,\infty}_{\mu} := L^{\infty}_{\mu}$.

Remark 3.4. See section 1.1 in [32] for more details about $[\cdot]_{\mu,p}$. We point out in particular that, as stated in Exercise 1.1.12 in that book, the space $L^{p,\infty}_{\mu}$ is metrizable for every $p > 0$ (and normable for $p > 1$, a fact that we are just about to use).

Markov’s classical inequality shows that if $p > 0$ is given, then for all $Y \in L^p_{\mu}$, $[Y]_{p,\mu} \leq \|Y\|_{p,\mu}$, so that L^p_{μ} is contained in the *weak L^p -space*, $L^{p,\infty}_{\mu}$. The inclusion $L^p_{\mu} \subset L^{p,\infty}_{\mu}$ is continuous.

Even more (see for instance [32], p.13, Exercise 1.1.12): if $p > 1$ there exists a norm, $||| \cdot |||_{p,\mu}$ on $L^{p,\infty}_{\mu}$ with respect to which $L^{p,\infty}_{\mu}$ is a Banach space, satisfying

$$[\cdot]_{p,\mu} \leq ||| \cdot |||_{p,\mu} \leq \frac{p}{p-1} [\cdot]_{p,\mu}.$$

In particular, if $p > 1$ and Y is any measurable function

$$(1 - \frac{1}{p}) |||Y|||_{p,\mu} \leq [Y]_{p,\mu} \leq \|Y\|_{p,\mu}, \quad (1.34)$$

(with the convention $\|Y\|_{p,\mu} = \infty$ if $Y \notin L_\mu^p$, and analogously if $Y \notin L_\mu^{p,\infty}$).

Let $T : L_\mu^1 \rightarrow L_\mu^1$ be a positive Dunford-Schwartz operator (Definitions 3.1 and 3.2) and define, for every $Y \in L_\mu^1$,

$$M_T Y := \sup_{n \in \mathbb{N}} \frac{1}{n} \left| \sum_{j=0}^{n-1} T^j Y \right|, \quad (1.35)$$

then (see [35], Lemma 6.1, p.51) for every $\alpha > 0$ the following Markov-type inequality holds

$$\mu([M_T |Y| > \alpha]) \leq \frac{1}{\alpha} E[|Y| I_{[M_T |Y| > \alpha]}] \leq \frac{1}{\alpha} E|Y|. \quad (1.36)$$

Therefore, for $Y \in L_\mu^{p,\infty}$

$$(1 - \frac{1}{p}) |||(M_T |Y|^p)^{1/p}|||_{p,\mu} \leq [(M_T |Y|^p)^{1/p}]_{p,\mu} := (\sup_{\alpha > 0} \alpha^p \mu[(M_T |Y|^p)^{\frac{1}{p}} > \alpha])^{1/p} \leq \|Y\|_{p,\mu},$$

where for the last inequality we used (1.36).

We summarize this discussion in the following proposition:

Proposition 3.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, let $T : L_\mu^1 \rightarrow L_\mu^1$ be a positive Dunford-Schwartz operator (Definitions 3.1 and 3.2), and define $M_T Y$ as in (1.35). Then for every $p > 1$ and every $Y \in L_\mu^{p,\infty}$*

$$|||(M_T |Y|^p)^{1/p}|||_{p,\mu} \leq \frac{p}{p-1} \|Y\|_{p,\mu}. \quad (1.37)$$

4 T -Filtrations and Adapted Processes

In this section we discuss the notions of T -filtrations and adapted processes. We shall also briefly discuss, in a heuristic language, how this notion codifies the idea of “initial conditions” for a given stationary process, an idea that will be formalized in a precise way and used in Chapter 3. The setting of adapted filtrations will be a fundamental part of the assumptions present along the main results of this work.

4.1 Definitions and Examples

Let us begin our discussion by giving the definition of a stationary process and an ergodic process, a family of processes for which the main results of this monograph are devoted.

Definition 4.1 (Stationary Processes, Ergodic Processes). *A stochastic process $(X_n)_{n \in \mathbb{Z}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called **stationary** if there exists an invertible, bimeasurable, measure-preserving transformation $T : \Omega \rightarrow \Omega$ such that for all $k \in \mathbb{Z}$, $X_k = T^k X_0$. The process is called **ergodic** if T is ergodic. If $X_0 \in L_\mathbb{P}^p$ (for some $p > 0$), we say that $(X_k)_{k \in \mathbb{Z}}$ is a p -**integrable process**.*

Let $(X_k)_{k \in \mathbb{Z}} = (T^k X_0)_{k \in \mathbb{Z}}$ be a stationary stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ (Definition 4.1), let $\mathcal{F}_0 \subset \mathcal{F}$ be a sub-sigma algebra of \mathcal{F} , and consider the following properties:

F1. T^{-1} is \mathcal{F}_0 measurable: $\mathcal{F}_0 \subset T^{-1}\mathcal{F}_0$, where for all $k \in \mathbb{Z}$

$$T^{-k}\mathcal{F}_0 := \{A \in \mathcal{F} : T^k A \in \mathcal{F}_0\} = \{T^{-k}B : B \in \mathcal{F}_0\}.$$

(here T^k denotes the k -fold composition of T).

F2. X_0 is \mathcal{F}_0 -measurable: $\sigma(X_0) \subset \mathcal{F}_0$.

Notice that if F1. is satisfied, the sequence $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ of sub-sigma algebras of \mathcal{F} defined by

$$\mathcal{F}_k := T^{-k}\mathcal{F}_0 \tag{1.38}$$

is nondecreasing ($\mathcal{F}_k \subset \mathcal{F}_{k+1}$ for all $k \in \mathbb{Z}$), and that X_k is \mathcal{F}_k -measurable if F2. holds.

Definition 4.2 (T -filtrations, adapted processes). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $T : \Omega \rightarrow \Omega$ be an invertible, bimeasurable, measure-preserving transformation.*

1. A **T -filtration** is a filtration of the form (1.38), where \mathcal{F}_0 satisfies F1.
2. The process $(T^k X_0)_{k \in \mathbb{Z}}$ is **adapted to** $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ if F2. holds.

Clearly, the “trivial” filtrations specified by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and by $\mathcal{F}_0 = \mathcal{F}$ are T -filtrations (any process defined in $(\Omega, \mathcal{F}, \mathbb{P})$ is adapted to the second one), thus the existence of these objects poses no serious questions.

Now note that the family of T -filtrations admits a partial order in the following way: given two T -filtrations $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ and $(\mathcal{G}_k)_{k \in \mathbb{Z}}$, $(\mathcal{F}_k)_{k \in \mathbb{Z}} \leq (\mathcal{G}_k)_{k \in \mathbb{Z}}$ if $\mathcal{F}_0 \subset \mathcal{G}_0$. With regards to this order, any stationary process $\mathbf{X} = (X_k)_{k \in \mathbb{Z}}$ admits a *minimal* (and unique) adapted filtration in the obvious way:

Definition 4.3 (Minimal Adapted Filtration). *Given a stationary process $(X_k)_{k \in \mathbb{Z}} = (T^k X_0)_{k \in \mathbb{Z}}$ (Definition 4.1), define \mathcal{M}_0 by*

$$\mathcal{M}_0 := \cap_{\alpha} \mathcal{G}_{\alpha} \tag{1.39}$$

where the intersection runs over the sub-sigma algebras $\mathcal{G}_{\alpha} \subset \mathcal{F}$ for which $\sigma(X_0) \subset \mathcal{G}_{\alpha} \subset T^{-1}\mathcal{G}_{\alpha}$. Then the filtration $(\mathcal{M}_k)_{k \in \mathbb{Z}} := (T^{-k}\mathcal{M}_0)_{k \in \mathbb{Z}}$ is the **minimal adapted filtration** associated to $(X_k)_{k \in \mathbb{Z}}$: it is the smallest T -filtration for which $(X_k)_{k \in \mathbb{Z}}$ is adapted (Definition 4.2).

To verify that $(\mathcal{M}_k)_{k \in \mathbb{Z}}$ is indeed a T -filtration notice the following: it is clear that if $\{\mathcal{G}_{\alpha}\}_{\alpha}$ is the family described in Definition 4.3 then

$$\mathcal{M}_0 \subset \cap_{\alpha} T^{-1}\mathcal{G}_{\alpha},$$

and note that for any given $A \in \cap_{\alpha} T^{-1}\mathcal{G}_{\alpha}$, if $A = T^{-1}A_{\alpha}$ ($A_{\alpha} \in \mathcal{G}_{\alpha}$), then $A_{\alpha} = TA$, which proves (A_{α}) does not depend on α) that

$$\cap_{\alpha} T^{-1}\mathcal{G}_{\alpha} \subset T^{-1} \cap_{\alpha} \mathcal{G}_{\alpha} =: T^{-1}\mathcal{M}_0.$$

The minimality of $(\mathcal{M}_k)_{k \in \mathbb{Z}}$ among the adapted filtrations and the uniqueness of \mathcal{M}_0 are clear from the definition.

Let us give a further definition, which we will need in subsequent sections.

Definition 4.4 (Left and Right sigma-algebras). *If $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ is a T -filtration (Definition 4.2), we define the **left** and **right** sigma algebras $\mathcal{F}_{-\infty}$, $\mathcal{F}_{+\infty}$ by*

$$\mathcal{F}_{-\infty} := \bigcap_{k \in \mathbb{Z}} \mathcal{F}_k \quad \text{and} \quad \mathcal{F}_{+\infty} := \sigma(\bigcup_{k \in \mathbb{Z}} \mathcal{F}_k). \quad (1.40)$$

To illustrate the notion of T -filtrations and adapted processes, it is convenient to look at an example, being perhaps the simplest nontrivial one that of *linear processes*.

Example 1 (Bernoulli Shifts and T -filtrations. Linear Processes). Consider the space $\Omega = \mathbb{C}^{\mathbb{Z}}$ and, for every $j \in \mathbb{Z}$, let $x_j : \Omega \rightarrow \mathbb{C}$ be the projection on the j -th coordinate: for every $\omega = (\omega_k)_{k \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$

$$x_j(\omega) = \omega_j. \quad (1.41)$$

Let \mathcal{F} be the sigma-algebra generated by the finite dimensional cylinders in Ω . This is, by sets of the form

$$H_{n,k,A} = \{\omega \in \Omega : (x_n(\omega), \dots, x_{n+k}(\omega)) \in A\} \quad (1.42)$$

where $(n, k) \in \mathbb{Z} \times \mathbb{N}$ and A is a Borel set in \mathbb{C}^{k+1} .

Given a sequence $(\xi_k)_{k \in \mathbb{N}}$ of random variables defined on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, consider the map $\xi : \Omega' \rightarrow \Omega$ given by $\xi(\omega') = (\xi_k(\omega'))_{k \in \mathbb{Z}}$. By the $\pi - \lambda$ theorem, there exists a unique probability measure \mathbb{P} in (Ω, \mathcal{F}) such that, for every set $H_{n,k,A}$ as in (4.16),

$$\mathbb{P}H_{n,k,A} = \mathbb{P}'\xi^{-1}H_{n,k,A}. \quad (1.43)$$

If the sequence $(\xi_k)_{k \in \mathbb{Z}}$ is stationary, in the sense that $\mathbb{P}'\xi^{-1}H_{n,k,A}$ is independent of n for every fixed k and A (for instance if $(\xi_k)_{k \in \mathbb{Z}}$ is i.i.d.), then the left shift $T : \Omega \rightarrow \Omega$, specified by $x_k(T\omega) = x_{k+1}(\omega)$ is an invertible bimeasurable, measure preserving transformation. In any case, if we define for every $k \in \mathbb{Z}$

$$\mathcal{F}_k := \sigma(x_j)_{j \leq k}. \quad (1.44)$$

then it is clear that $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ is a T -filtration.

Note that, in this setting, the sequence of coordinate functions $(x_k)_{k \in \mathbb{Z}}$ is a copy (in distribution) of $(\xi_k)_{k \in \mathbb{Z}}$, thus we can replace “ ξ_k ” by “ x_k ” when referring to issues about distribution.

Assume, in addition to stationarity, that $x_0 \in L_{\mathbb{P}}^2$ (therefore $x_k \in L_{\mathbb{P}}^2$ for every $k \in \mathbb{Z}$), that $Ex_0 = 0$ and that the x_k 's are orthogonal:

$$E[x_k \overline{x_l}] = \delta_{k,l} E[|x_0|^2]$$

where $\delta_{k,l}$ is the Kronecker δ -function ($\delta_{k,l} \in \{0, 1\}$, and $\delta_{k,l} = 0$ if and only if $k \neq l$).

In this setting, given any sequence $(a_k)_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$ and any $k \in \mathbb{Z}$, the function

$$X_k(\omega) = \sum_{j \in \mathbb{Z}} a_j x_{k-j}(\omega) \quad (1.45)$$

is well defined in the $L^2_{\mathbb{P}}$ sense: for any $k \in \mathbb{Z}$ and $N \in \mathbb{N}$, if $J \subset \mathbb{Z}$ is a finite set with $[-N, N] \cap \mathbb{Z} \subset J$ and $J' := J \setminus [-N, N]$

$$E\left[\left|\sum_{j \in J'} a_j x_{k-j}\right|^2\right] \leq E[|x_0|^2] \sum_{|j| > N} |a_j|^2.$$

which guarantees the convergence (well definition) of X_k because $\sum_{j \in \mathbb{Z}} |a_j|^2 < \infty$ ($L^2_{\mathbb{P}}$ is complete). Note also that for every $k \in \mathbb{Z}$, $X_k = T^k X_0$, and that $(X_k)_{k \in \mathbb{Z}}$ is $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ -adapted provided that $a_j = 0$ for every $j < 0$. We will explore a particular case of this example when proving Theorem 16.3 in Chapter 4.

Remark 4.1. Note that, by replacing a_k by $a'_k = E[|x_0|^2] a_k$ and x_k by $x_k / \|x_k\|_{\mathbb{P}, 2}$, we can assume without loss of generality that $E[|x_k|^2] = 1$.

4.2 Heuristic Interpretation

In a heuristic language, taking \mathcal{F}_0 as the “information available to an observer”, we require that T preserves the information in \mathcal{F}_0 in order to obtain a T -filtration: any set of the form TA for $A \in \mathcal{F}_0$ *still* belongs to \mathcal{F}_0 . Pulling this heuristic further, we can thus think of \mathcal{F}_0 as the *deterministic part* of the dynamical system $T : \Omega \rightarrow \Omega$: an observer capable of knowing all the elements in \mathcal{F}_0 can follow their evolution under T in a deterministic way.

It is also usual to interpret \mathcal{F}_0 as “the information from the past”, an interpretation that is particularly meaningful in the case of linear processes or, more generally, in the setting of functions of stationary Markov Chains (see Example 6). Example 1 allows us to see how this naturally makes sense: \mathcal{F}_0 , in this case, is the sigma algebra generated by all the coordinates of the process up to the time $k = 0$.

Now, a T -filtration is adapted to $(X_k)_{k \in \mathbb{Z}} = (T^k X_0)_{k \in \mathbb{Z}}$ if the information provided by X_0 is deterministic: the observer is able to know the outcome of the process at the time $k = 0$. In this setting we can think of an adapted process as a process “with given initial conditions”: the outcome of X_0 is known at the moment of running the process.

How does this knowledge affect the asymptotics related to $(X_k)_{k \in \mathbb{Z}}$? This is, in broad terms, the question addressed by the notion of *quenched convergence*, to be discussed in Chapter 3. In short, and following the traditional interpretation of “conditioning”, we will codify the “influence” of this knowledge by means of the conditional expectation with respect to \mathcal{F}_0 .

4.3 Interactions with the Koopman Operator

To begin with this section let us prove the following result.

Proposition 4.1. *Let $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ be a T -filtration (Definition 4.2), let $T : L_{\mathbb{P}}^1 \rightarrow L_{\mathbb{P}}^1$ be the corresponding Koopman operator (Definition 1.2) and for every $k \in \mathbb{Z}$, denote by E_k the conditional expectation with respect to \mathcal{F}_k : for every $Y \in L_{\mathbb{P}}^1$ and $k \in \mathbb{Z}$:*

$$E_k Y := E[Y | \mathcal{F}_k]. \quad (1.46)$$

Then for every $k, r \in \mathbb{Z}$

$$T^r E_k = E_{k+r} T^r \quad (1.47)$$

as operators in $L_{\mathbb{P}}^1$.

Proof: Given $Y \in L_{\mathbb{P}}^1$, since clearly $T^r E_k Y$ is \mathcal{F}_{r+k} -measurable, we need to prove that for all $A \in \mathcal{F}_{k+r}$,

$$E[(T^r E_k Y) I_A] = E[(T^r Y) I_A].$$

To do so, let $A' = T^{r+k} A$. Notice that $A' \in \mathcal{F}_0$, and therefore $T^{-k} A' \in \mathcal{F}_k$. Using this and the fact that T is measure preserving we see that

$$\begin{aligned} E[(T^r E_k Y) I_A] &= E[T^r [(E_k Y) I_{T^{-k} A'}]] = E[E_k [Y I_{T^{-k} A'}]] = E[Y I_{T^{-k} A'}] \\ &= E[(T^r Y) I_{T^{-(r+k)} A'}] = E[(T^r Y) I_A] \end{aligned}$$

as desired. \square

Now notice the following: assume that, for a given $k \in \mathbb{N}$, $(E_0 T)^k = E_0 T^k$, then:

$$(E_0 T)^{k+1} = E_0 T (E_0 T)^k = E_0 T E_0 T^k = E_0 E_1 T^{k+1} = E_0 T^{k+1}$$

which shows, by induction on k , that for every $k \in \mathbb{N}^*$,

$$(E_0 T)^k = E_0 T^k \quad (1.48)$$

The operator $E_0 T$ satisfies the following ergodic theorem.

Theorem 4.1 (An Ergodic Theorem for Adapted T -filtrations). *In the context of Theorem 3.2 and Proposition 4.1, given $p \geq 1$ and $Y \in L_{\mathbb{P}}^p$:*

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} E_0 T^k Y e^{ik\theta} = E_0 P_{\theta} Y, \quad \mathbb{P}\text{-a.s and in } L_{\mathbb{P}}^p. \quad (1.49)$$

Proof: Let p and Y be as in the given hypothesis. The convergence \mathbb{P} -a.s and in $L_{\mathbb{P}}^p$ follows via the following argument, similar to the one given for the proof of Theorem 3.2. The details are left to the reader.

Convergence. With the notation introduced in Definition 3.3 and the discussion following it, and defining

$$\tilde{E}_0 := E[\cdot | \mathcal{B} \otimes \mathcal{F}_0]$$

(where the conditional expectation is with respect to $\lambda \times \mathbb{P}$), we can observe that (1.48) holds with \tilde{E}_0 in place of E_0 and \tilde{T}_θ in place of T . An application of Theorem 3.1 to the positive Dunford-Schwartz operator $\tilde{E}_0 \tilde{T}_\theta$ allows one to see that

$$\frac{1}{n} \sum_{k=0}^{n-1} \tilde{E}_0 \tilde{T}_\theta^k \tilde{Y} = \frac{1}{n} \sum_{k=0}^{n-1} \tilde{E}_0 \widetilde{e^{ik\theta} T^k Y} \rightarrow_n P_{\tilde{E}_0 \tilde{T}_\theta} \tilde{Y},$$

$\lambda \times \mathbb{P}$ -a.s. and in $L_{\lambda \times \mathbb{P}}^p$. This implies that there exists a function⁸ $P_{E_0, T, \theta} Y : \Omega \rightarrow \mathbb{C}$ such that

$$\frac{1}{n} \sum_{k=0}^{n-1} E_0 T^k Y e^{ik\theta} \rightarrow_n P_{E_0, T, \theta} Y$$

\mathbb{P} -a.s. and in $L_{\mathbb{P}}^p$.

Limit function. To identify $P_{E_0, T, \theta} Y$ we use the continuity of E_0 as a linear operator in $L_{\mathbb{P}}^p$: since, according to Theorem 3.2, $S_n(Y, \theta) \rightarrow P_\theta Y$ in $L_{\mathbb{P}}^p$ as $n \rightarrow \infty$, $E_0 S_n(Y, \theta)$ converges in $L_{\mathbb{P}}^p$ as $n \rightarrow \infty$, and

$$\lim_n E_0 [S_n(Y, \theta)] = E_0 [\lim_n S_n(Y, \theta)] = E_0 P_\theta Y,$$

as claimed. \square

Corollary 4.2. *In the context of Theorem 4.1. If T is ergodic,*

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} E_0 T^k Y = E[Y], \quad \mathbb{P}\text{-a.s. and in } L_{\mathbb{P}}^p. \quad (1.50)$$

Proof: Immediate from Remark 3.3 and the fact that, in this case, $P_0 Y = EY$, \mathbb{P} -a.s. \square

5 The Autocovariance Function and the Spectral Density

In this section we discuss the notions of the *autocovariance function* and the *spectral density* of a stationary process.

The autocovariance function is of relevance both in the theoretical and applied aspects of the theory of stochastic processes because it encodes the covariance structure of a given process (allowing inferences about, for instance, rates of decay), and its estimation is part of the study carried out here.

To give a method to estimate the values of the autocovariance function we will introduce the closely related notion of the spectral density, which can be computed studying the asymptotic behavior of the normalized L^2 -norms of the discrete Fourier transforms (see Theorem 5.4 below).

⁸Actually given by $\omega \mapsto e^{-iu} P_{\tilde{E}_0 \tilde{T}_\theta} \tilde{Y}(u, \omega)$, but this is not the representation that we are looking for.

5.1 The Autocovariance Function

To make the discussion clear let us start by recalling the following definition.

Definition 5.1 (Nonnegative Definite Function). *A function $\gamma : \mathbb{Z} \rightarrow \mathbb{C}$ is **nonnegative definite** if for all vectors $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{C}^n$*

$$\sum_{i,j=1}^n c_i \gamma(i-j) \bar{c}_j \geq 0. \quad (1.51)$$

These functions happen to be an important object in the study of the spectral properties of stationary sequences. The essential connection with this topic is *Herglotz's Theorem*:

Theorem 5.1 (Herglotz's Theorem). *A function $\gamma : \mathbb{Z} \rightarrow \mathbb{C}$ is nonnegative definite (Definition 5.1) if and only if there exists a nondecreasing, right continuous bounded function $F : [-\pi, \pi) \rightarrow [0, +\infty)$ such that $F(0) = 0$ and*

$$\gamma(n) = \int_{-\pi}^{\pi} e^{in\theta} d\mu_F(\theta)$$

where μ_F denotes the measure induced by F : $\mu_F((a, b]) := F(b) - F(a)$ for all $[a, b] \subset [-\pi, \pi)$.

Proof: This is Theorem 4.3.1 in [14]. □

Remark 5.1. By the periodicity of the functions $\theta \mapsto e^{in\theta}$, the conclusion of this theorem remains valid if we substitute $[-\pi, \pi)$ by any interval of length 2π . If we consider for instance the interval $[0, 2\pi)$ with the Borel sigma-algebra \mathcal{B} and the measure

$$\mu'_F(A) = \mu_F(A \cap [0, \pi)) + \mu_F((A - 2\pi) \cap [-\pi, 0))$$

for all $A \in \mathcal{B}$ (where $A - 2\pi := \{a - 2\pi : a \in A\}$), then the statement of Theorem 5.1 remains valid replacing $[-\pi, \pi)$ by $[0, 2\pi)$ and μ_F by μ'_F .

The connection of this theorem with the theory of stationary stochastic processes is made via the notion of the *autocovariance function*.

Definition 5.2 (Autocovariance Function). *Given a stationary square-integrable process $(X_n)_{n \in \mathbb{Z}}$ (Definition 4.1), we define the **autocovariance function** $\gamma : \mathbb{Z} \rightarrow \mathbb{C}$ by*

$$\gamma(n) := E[(X_0 - EX_0)(\bar{X}_n - \overline{EX_n})] = E[(X_0 - EX_0)(\bar{X}_n - \overline{EX_0})]. \quad (1.52)$$

Remark 5.2. Note that $\gamma(\cdot)$ encodes all the covariances of the process $(X_n)_{n \in \mathbb{Z}}$: given integers j, k

$$\begin{aligned} E[(X_j - EX_j)(\bar{X}_k - \overline{EX_k})] &= E[T^j[(X_0 - EX_0)(\bar{X}_{k-j} - \overline{EX_0})]] = \\ &= E[(X_0 - EX_0)(\bar{X}_{k-j} - \overline{EX_0})] = \gamma(k-j). \end{aligned}$$

Note also that the autocovariance function is hermitian ($\gamma(n) = \overline{\gamma(-n)}$) and nonnegative definite (Definition 1.51): given $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{C}^n$, and denoting by $\mathbb{X}_n = (X_1 - EX_1, \dots, X_n - EX_n)$

$$\sum_{i,j} c_i \gamma(i-j) \bar{c}_j = E\left[\sum_{i,j} c_i (X_i - EX_i) (\overline{X_j - EX_j}) \bar{c}_j\right] = E[|\mathbf{c} \cdot \mathbb{X}_n|^2] \geq 0$$

Herglotz's theorem implies therefore the following.

Proposition 5.1 (Existence of the Spectral Measure). *Given a stationary square-integrable process $\mathbf{X} = (X_k)_{k \in \mathbb{Z}}$ (Definition 4.1) there exists a (finite) measure $m_{\mathbf{X}}$ on $([0, 2\pi), \mathcal{B})$ such that the autocovariance function $\gamma : \mathbb{Z} \rightarrow \mathbb{C}$ of $(X_k)_{k \in \mathbb{Z}}$ (Definition 5.2) is given by*

$$n \mapsto \gamma(n) = \int_0^{2\pi} e^{in\theta} dm_{\mathbf{X}}(\theta). \quad (1.53)$$

Proof: Use Herglotz's theorem (Theorem 5.1) together with Remark 5.1. \square

5.2 The Spectral Density

Our goal in this section is to connect the notion of the *spectral density* of a stationary process with the asymptotic theory of discrete Fourier transforms. To begin with, let us start by recalling the *Féjer-Lebesgue Theorem*.

Theorem 5.2 (Féjer-Lebesgue). *Let $f \in L^1_{\lambda}$ be given and denote by \hat{f} the Fourier transform of f (Definition 2.4). Then the sequence of functions $(C_n f)_{n \geq 0}$ defined by*

$$C_n f(\theta) := \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=-j}^j \hat{f}(k) e^{ik\theta} \quad (1.54)$$

converges to f λ -a.s.

Proof: See the proof of Theorem 3.3.3 in [32]. \square

Remark 5.3 (L^p_{λ} convergence in Theorem 5.2). According to the referred proof in [32], the convergence in (1.54) holds in the L^p_{λ} sense if $p > 1$ and $f \in L^p_{\lambda}$: in such case there exists a constant C_p such that

$$\| \sup_{n \in \mathbb{N}^*} |C_n f| \|_{\lambda, p} \leq C_p \|f\|_{\lambda, p}.$$

In particular, $(C_n f - f)_{n \in \mathbb{N}^*}$ is dominated in L^p_{λ} (by $2 \sup_{n \in \mathbb{N}^*} |C_n f|$), and the dominated convergence theorem implies that $\|C_n f - f\|_{\lambda, p} \rightarrow 0$ as $n \rightarrow \infty$.

We saw in Proposition 5.1 that for a stationary square-integrable process $\mathbf{X} = (X_k)_{k \in \mathbb{Z}}$ there exists a measure $m_{\mathbf{X}}$ on $([0, 2\pi), \mathcal{B})$ such that

$$E[(X_0 - EX_0)(\overline{X_k - EX_k})] = \int_0^{2\pi} e^{ik\theta} dm_{\mathbf{X}}(\theta) \quad (1.55)$$

for all $k \in \mathbb{Z}$.

Note that, if F is absolutely continuous with respect to λ and

$$f(\theta) := \frac{dm_{\mathbf{x}}}{d\lambda}(\theta) \quad (1.56)$$

is the Radon-Nikodym derivative of $m_{\mathbf{x}}$ with respect to λ , then it follows from (1.55) that, if we denote by \hat{f} the Fourier transform of f , then

$$\hat{f}(-k) = \int_0^{2\pi} f(\theta) e^{ik\theta} d\lambda(\theta) = E[(X_0 - EX_0)(\overline{X_k} - \overline{EX_0})] = \gamma(k)$$

This is: the autocovariance function of $(X_k)_{k \in \mathbb{Z}}$ is given by the sequence of the negative Fourier coefficients of f . This justifies the following definition.

Definition 5.3 (Spectral Density). *We say that a stationary square-integrable process $(X_k)_{k \in \mathbb{Z}}$ (Definition 4.1) admits a **spectral density** if there exists a nonnegative function $f \in L^1_{\lambda}([0, 2\pi))$ such that for every $k \in \mathbb{Z}$*

$$\hat{f}(-k) = \gamma(k), \quad (1.57)$$

where \hat{f} denotes the Fourier transform of f (Definition 2.4) and γ is the autocovariance function of $(X_k)_{k \in \mathbb{Z}}$ (Definition 5.2).

Remark 5.4. It is an immediate consequence of Theorem 5.2 that if $(X_k)_{k \in \mathbb{Z}}$ admits a spectral density f then it is unique (up to a set of λ -measure zero). Note also that if the process $(X_k)_{k \in \mathbb{Z}}$ is real-valued and admits a spectral density f , then $\hat{f}(k) = \gamma(k)$ for all $k \in \mathbb{Z}$ (γ is hermitian and real valued, i.e., even).

5.3 Regular Processes

Let us make now a short digression that will allow us to relate the notion of T -filtrations to the existence of the spectral density.

Definition 5.4 (Regularity of an Adapted Process). *Let $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ be a T -filtration and let $(X_k)_{k \in \mathbb{Z}} = (T^k X_0)_{k \in \mathbb{Z}}$ be a $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ -adapted stationary square-integrable process (Definition 4.2). The process is called **regular** (with respect to $(\mathcal{F}_k)_{k \in \mathbb{Z}}$) if $E[X_0 | \mathcal{F}_{-k}]$ converges to 0 in $L^2_{\mathbb{P}}$. This is, if*

$$\lim_{k \rightarrow \infty} E[|E[X_0 | \mathcal{F}_{-k}]|^2] = 0. \quad (1.58)$$

Remark 5.5. Recall the notation introduced in Definition 4.4 and Proposition 4.1. Since for every $p \geq 1$, $k \in \mathbb{Z}$ and $Y \in L^p_{\mathbb{P}}$ the process $(E_{-j+k} Y)_{j \geq 0}$ is a reverse martingale in $L^p_{\mathbb{P}}$, the reverse martingale convergence theorem (see Theorem 5.6.1 and Exercise 5.6.1 in [25]) and the continuity of the Koopman operator T imply that the following equalities hold both \mathbb{P} -a.s and in $L^p_{\mathbb{P}}$:

$$T^k E_{-\infty} Y = T^k \lim_{j \rightarrow \infty} E_{-j+k} Y = \lim_{j \rightarrow \infty} T^k E_{-j+k} Y = \lim_{j \rightarrow \infty} E_{-j} T^k Y = E_{-\infty} T^k Y. \quad (1.59)$$

In particular, (1.58) is equivalent to the following condition: for every $k \in \mathbb{Z}$

$$E[X_k | \mathcal{F}_{-\infty}] = 0 \quad (\mathbb{P}\text{-a.s.}). \quad (1.60)$$

Now notice the following: if $(\mathcal{M}_k)_{k \in \mathbb{Z}}$ is the minimal adapted filtration associated to $(X_k)_{k \in \mathbb{Z}}$ (Definition 4.3) and $\mathcal{M}_{-\infty}$ is its left sigma algebra of this filtration (Definition 4.4) then, since (1.60) is equivalent to the condition $E[X_k I_A] = 0$ for every $A \in \mathcal{F}_{-\infty}$ and $\mathcal{M}_{-\infty} \subset \mathcal{F}_{-\infty}$, we have that if $(X_k)_{k \in \mathbb{Z}}$ is regular with respect to $(\mathcal{F}_k)_{k \in \mathbb{Z}}$, then for every $k \in \mathbb{Z}$

$$E[X_k | \mathcal{M}_{-\infty}] = 0. \quad (1.61)$$

In virtue of Remark 5.5 this gives the following result.

Proposition 5.2 (Regularity and Minimal Adapted Filtrations). *If a process $(X_k)_{k \in \mathbb{Z}}$ is regular with respect to some (adapted) filtration $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ (Definition 5.4), then it is regular with respect to its minimal adapted filtration $(\mathcal{M}_k)_{k \in \mathbb{Z}}$ (Definition 4.3).*

Since the minimal adapted filtration is unique, this shows that the notion of regularity of an adapted process can be made “unambiguous” if we declare a process “regular” if it is regular with respect to its minimal adapted filtration.

5.4 On the Existence of the Spectral Density

What stationary processes $\mathbf{X} = (X_k)_{k \in \mathbb{Z}}$ admit a spectral density? First, as stated by Theorem 31.28 in [11], the existence of f is equivalent to the *absolute continuity* (in the sense of real calculus) of the distribution function $F_{\mathbf{X}}$ of $m_{\mathbf{X}}$ ($F_{\mathbf{X}}(t) := m_{\mathbf{X}}((-\infty, t])$): f exists if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\{[a_k, b_k]\}_{k=1}^n$ is any collection of disjoint intervals contained in $[0, 2\pi)$ for which $\sum_{k=1}^n (b_k - a_k) < \delta$, then $\sum_{k=1}^n (F_{\mathbf{X}}(b_k) - F_{\mathbf{X}}(a_k)) < \epsilon$.

An interesting question is how to characterize the existence of the spectral density in terms of rates of decay of the values of the autocovariance function γ .

To be more precise, note first that by the proof of Proposition 5.1, $m_{\mathbf{X}}$ is characterized by the equation (1.53).

Now, as proved in [14], Corollary 4.3.1 (together with Remark 5.1 above), *every* function $\gamma : \mathbb{Z} \rightarrow \mathbb{C}$ that can be represented in the form

$$\gamma(k) = \int_0^{2\pi} e^{ik\theta} d\mu(\theta) \quad (1.62)$$

for some finite measure μ on $([0, 2\pi), \mathcal{B})$, is the autocovariance function of some stationary square-integrable process. Note again that $\gamma(-k)$ is (by definition) the k -th Fourier coefficient of the measure μ .

Thus, since any finite measure on $([0, 2\pi), \mathcal{B})$ is determined by the sequence of its Fourier coefficients ([8], Proposition 6.3), the problem of the absolute continuity of $m_{\mathbf{X}}$ (for any \mathbf{X}) is equivalent to the following question:

Question: *Let μ be a finite measure on $[0, 2\pi)$ and let $\gamma : \mathbb{Z} \rightarrow \mathbb{C}$ be given by (1.62). What conditions on the sequence $(\gamma(k))_{k \in \mathbb{Z}}$ are necessary and/or sufficient to guarantee that μ is absolutely continuous with respect to λ ?*

Any answer to this question has an immediate translation to a criterion about the existence of the spectral density of a stationary process $(X_k)_{k \in \mathbb{Z}}$ in terms of the sequence of its covariances $(\gamma(k))_{k \in \mathbb{Z}} = (E[(X_0 - EX_0)(\overline{X_k} - \overline{EX_0})])_{k \in \mathbb{Z}}$.

The following criterion is just a reformulation of one of the equivalences of Theorem 1 in [37].

Theorem 5.3 (Absolute Continuity via Fourier Coefficients). *Let μ be a finite measure in $[0, 2\pi)$ and define $\gamma : \mathbb{Z} \rightarrow \mathbb{C}$ by (1.62). Then the following are equivalent*

1. μ is absolutely continuous with respect to λ .
2. There exists a sequence of complex numbers $(a_k)_{k \in \mathbb{Z}} \in l^2_{\mathbb{Z}}$ such that for all $n \in \mathbb{Z}$

$$\gamma(n) = \sum_{j \in \mathbb{Z}} a_j \overline{a_{j+n}}. \quad (1.63)$$

Remark 5.6. The convolution $a * b$ between two sequences $a = (a_k)_{k \in \mathbb{Z}}$ and $b = (b_k)_{k \in \mathbb{Z}}$ in $l^2(\mathbb{Z})$ is equal to the sequence $a * b = ((a * b)(k))_{k \in \mathbb{Z}} \in l^1(\mathbb{Z})$ given by

$$(a * b)(k) = \sum_{j \in \mathbb{Z}} a_j b_{j-k}.$$

Using the fact that every function $f \in L^1_{\lambda}$ is the product of two functions in L^2_{λ} (consider a branch of the square root), it is possible to show that the correspondence $l^2(\mathbb{Z}) \times l^2(\mathbb{Z}) \rightarrow L^1_{\lambda}$ given by $(a, b) \mapsto f_{(a,b)}$ where

$$f_{(a,b)}(\theta) := \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=-k}^k (a * b)(j) e^{ik\theta}$$

is surjective (the limit is λ -a.e well defined by Theorem 5.2). In this language, Theorem 5.3 can be understood as the statement that every absolutely continuous measure in \mathbb{T} corresponds to a sequence of the form $a * \overline{a}$ for some $a \in l^2(\mathbb{Z})$, where $\overline{a} = (\overline{a_k})_{k \in \mathbb{Z}}$ is the conjugate sequence of a . The corresponding density is actually given by $f_{(a,\overline{a})}$, which is the same as the function

$$\theta \mapsto \left| \sum_{k \in \mathbb{Z}} a_k e^{ik\theta} \right|^2.$$

In the context of Theorem 5.3, it is possible to give other sufficient conditions implying the absolute continuity of μ with respect to λ . Assume for instance that $(\gamma(k))_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$. Then by Proposition 2.2 the function

$$f(\theta) = \sum_{k \in \mathbb{Z}} \gamma(k) e^{ik\theta}$$

is the spectral density of $(X_k)_{k \in \mathbb{Z}}$. In particular, the condition

$$\sum_{k \geq 0} |E[(X_0 - EX_0)(\overline{X_k} - \overline{EX_0})]|^2 < \infty \quad (1.64)$$

implies the existence of the spectral density. For other sufficient conditions see for instance [19] and the references therein.

Our next result is the following:

Proposition 5.3 (Spectral Density via Regularity). *Every regular process (Definition 5.4) admits a spectral density.*

Proof:⁹ With the notation (1.46) define, for every $k \in \mathbb{Z}$ and $Y \in L_{\mathbb{P}}^1$,

$$\mathcal{P}_k Y := (E_k - E_{k-1})Y. \quad (1.65)$$

Remember now that $L_{\mathbb{P}}^2(\mathcal{F}_k) \subset L_{\mathbb{P}}^2$ denotes the subspace of functions that are measurable with respect to \mathcal{F}_k , and denote also by

$$V_k := L_{\mathbb{P}}^2(\mathcal{F}_k) \ominus L_{\mathbb{P}}^2(\mathcal{F}_{k-1}) \quad (1.66)$$

the orthogonal complement of $L_{\mathbb{P}}^2(\mathcal{F}_{k-1})$ in $L_{\mathbb{P}}^2(\mathcal{F}_k)$. Then, by defining $\mathcal{F}_{-\infty}$ as in (1.40), we see that

$$L_{\mathbb{P}}^2(\mathcal{F}_k) \ominus L_{\mathbb{P}}^2(\mathcal{F}_{-\infty}) = \bigoplus_{j \leq k} V_j \quad (1.67)$$

and that \mathcal{P}_k , restricted to $L_{\mathbb{P}}^2$, is just the orthogonal projection on the space V_k . In particular, since under (1.58), $X_k \in L_{\mathbb{P}}^2(\mathcal{F}_k) \ominus L_{\mathbb{P}}^2(\mathcal{F}_{-\infty})$ (see Remark 5.5), we have that for every $k \in \mathbb{Z}$

$$X_k = \sum_{j \in \mathbb{Z}} \mathcal{P}_{-j} X_k = \sum_{j \geq -k} \mathcal{P}_{-j} X_k$$

and therefore, by orthogonality and Proposition 4.1,

$$E[|X_0|^2] = \sum_{k \geq 0} E[|\mathcal{P}_{-k} X_0|^2] = \sum_{k \geq 0} E[|\mathcal{P}_0 X_k|^2]. \quad (1.68)$$

It follows from Proposition 2.3 that the function $f : [0, 2\pi) \rightarrow [0, \infty)$ specified by

$$f(\theta) = E\left[\left|\sum_{k \geq 0} \mathcal{P}_0 X_k e^{ik\theta}\right|^2\right]$$

is well defined. More precisely: for λ -a.e θ the integrand converges \mathbb{P} -a.s and the integral (with respect to \mathbb{P}) makes sense.

We claim that f is the spectral density of $(X_k)_{k \in \mathbb{Z}}$.

Fix $k \in \mathbb{Z}$ and begin by noticing that, by orthogonality and Proposition 4.1,

$$\begin{aligned} E[X_0 \overline{X_{-k}}] &= E\left[\left(\sum_{j \in \mathbb{Z}} \mathcal{P}_{-j} X_0\right) \overline{\left(\sum_{l \in \mathbb{Z}} \mathcal{P}_{-l} X_{-k}\right)}\right] = \sum_{j \geq 0} E[(\mathcal{P}_{-j} X_0) \overline{(\mathcal{P}_{-j} X_{-k})}] = \\ &= \sum_{j \geq 0} E[(\mathcal{P}_0 X_j) \overline{(\mathcal{P}_0 X_{j-k})}]. \end{aligned} \quad (1.69)$$

⁹This argument follows the proof of Theorem 3 in [37].

Our goal is thus to prove that f is integrable and that for every $k \in \mathbb{Z}$, $\hat{f}(k)$ is equal to the last term in (1.69).

To begin with, define Ω_1 as the set of probability one

$$\Omega_1 := \{\omega \in \Omega : \sum_{k \geq 0} |\mathcal{P}_0 X_k(\omega)|^2 < \infty\}. \quad (1.70)$$

1. *The function f is integrable.* By (1.68) and Carleson's Theorem (Theorem 2.1) the function

$$\theta \mapsto \sum_{k \geq 0} \mathcal{P}_0 X_k(\omega) e^{ik\theta}$$

is well defined (the series converges λ -a.s) for every $\omega \in \Omega_1$. It follows from the dominated convergence theorem and (1.18) (see also the line following that equation) that for every $\omega \in \Omega_1$

$$\int_0^{2\pi} \left| \sum_{k \geq 0} \mathcal{P}_0 X_k(\omega) e^{ik\theta} \right|^2 d\lambda(\theta) = \sum_{k \geq 0} |\mathcal{P}_0 X_k(\omega)|^2 < \infty.$$

Now, by Tonelli's theorem, Proposition 4.1 and the monotone convergence theorem

$$\int_0^{2\pi} f(\theta) d\lambda(\theta) = E \left[\int_0^{2\pi} \left| \sum_{k \geq 0} \mathcal{P}_0 X_k e^{ik\theta} \right|^2 d\lambda(\theta) \right] = \sum_{k \geq 0} E[|\mathcal{P}_0 X_k|^2] =$$

$$\sum_{k \geq 0} E[|\mathcal{P}_{-k} X_0|^2] = E[|X_0|^2],$$

which shows that $f \in L^1_\lambda$. Note also that this proves the required equality $\hat{f}(0) = E[|X_0|^2]$.

2. *The Fourier coefficient $\hat{f}(k)$ is given by the last term of (1.69).* Fix $k \in \mathbb{Z}$ and note first that, by (1.18) and the dominated convergence theorem, the following holds: for every $\omega \in \Omega_1$,

$$\begin{aligned} \int_0^{2\pi} e^{-ik\theta} \left| \sum_{j \geq 0} \mathcal{P}_0 X_j(\omega) e^{ij\theta} \right|^2 d\lambda(\theta) &= \lim_N \int_0^{2\pi} e^{-ik\theta} \left| \sum_{j=0}^N \mathcal{P}_0 X_j(\omega) e^{ij\theta} \right|^2 d\lambda(\theta) = \\ \lim_N \int_0^{2\pi} \sum_{j=0}^N \sum_{l=0}^N (\mathcal{P}_0 X_j(\omega)) (\overline{\mathcal{P}_0 X_l(\omega)}) e^{i(j-l-k)\theta} d\lambda(\theta) &= \lim_N \sum_{j=0}^N (\mathcal{P}_0 X_j) (\overline{\mathcal{P}_0 X_{j-k}}) = \\ \sum_{j \geq 0} (\mathcal{P}_0 X_j(\omega)) (\overline{\mathcal{P}_0 X_{j-k}(\omega)}). \end{aligned}$$

Now notice the following: by the Cauchy-Schwartz inequality and Proposition 4.1,

$$\sum_{j \geq 0} E[|(\mathcal{P}_0 X_j) (\overline{\mathcal{P}_0 X_{j-k}})|] \leq \sum_{j \geq 0} (E[|\mathcal{P}_0 X_j|^2])^{1/2} (E[|\mathcal{P}_0 X_{j-k}|^2])^{1/2} \leq$$

$$\begin{aligned}
& \left(\sum_{j \geq 0} E[|\mathcal{P}_0 X_j|^2] \right)^{1/2} \left(\sum_{l \geq 0} E[|\mathcal{P}_0 X_{l-k}|^2] \right)^{1/2} = \left(\sum_{j \geq 0} E[|\mathcal{P}_0 X_j|^2] \right)^{1/2} \left(\sum_{l \geq -k} E[|\mathcal{P}_0 X_l|^2] \right)^{1/2} = \\
& \left(\sum_{j \geq 0} E[|\mathcal{P}_{-j} X_0|^2] \right)^{1/2} \left(\sum_{l \geq -k} E[|\mathcal{P}_{-l} X_0|^2] \right)^{1/2} = \sum_{j \geq 0} E[|\mathcal{P}_0 X_j|^2] = E[|X_0|^2],
\end{aligned}$$

and therefore, by the dominated convergence theorem

$$E\left[\sum_{j \geq 0} (\mathcal{P}_0 X_j(\omega)) \overline{(\mathcal{P}_0 X_{j-k}(\omega))}\right] = \sum_{j \geq 0} E[(\mathcal{P}_0 X_j(\omega)) \overline{(\mathcal{P}_0 X_{j-k}(\omega))}]. \quad (1.71)$$

This information allows us to finish the proof: an application of Fubini's theorem and (1.71) gives

$$\begin{aligned}
\hat{f}(k) &:= \int_0^{2\pi} f(\theta) e^{-ik\theta} d\lambda(\theta) = \int_0^{2\pi} E\left[\left|\sum_{j \geq 0} \mathcal{P}_0 X_j e^{ij\theta}\right|^2\right] e^{-ik\theta} d\lambda(\theta) = \\
& \int_0^{2\pi} E\left[\left(\sum_{j \geq 0} \mathcal{P}_0 X_j e^{ij\theta}\right) \left(\sum_{l \geq 0} \overline{\mathcal{P}_0 X_l} e^{-il\theta}\right)\right] e^{-ik\theta} d\lambda(\theta) = \\
& E\left[\int_0^{2\pi} \sum_{j, l \geq 0} (\mathcal{P}_0 X_j) \overline{(\mathcal{P}_0 X_l)} e^{i(j-l-k)\theta} d\lambda(\theta)\right] = \\
& E\left[\sum_{k \geq 0} (\mathcal{P}_0 X_j) \overline{(\mathcal{P}_0 X_{j-k})}\right] = \sum_{k \geq 0} E[(\mathcal{P}_0 X_j) \overline{(\mathcal{P}_0 X_{j-k})}].
\end{aligned}$$

which is the desired expression. \square

Remark 5.7. Note that the function

$$D_0(\theta) = \sum_{k \geq 0} \mathcal{P}_0 X_k e^{ik\theta} \quad (1.72)$$

is an adapted martingale difference with respect to $(\mathcal{F}_k)_{k \geq 0}$: $D_0(\theta)$ is \mathcal{F}_0 -measurable and, for every $k \geq 1$, $E_{k-1} T^k D_0(\theta) = T^k E_{-1} D_0(\theta) = 0$. In subsequent proofs we will show that if for every $n \geq 0$ we define

$$M_n(\theta) := \sum_{k=0}^{n-1} T^k D_0(\theta) e^{ik\theta}, \quad (1.73)$$

then the quenched asymptotics of $(M_n(\theta))_{n \geq 0}$ can be transported to corresponding results for $(S_n(\theta) - E_0 S_n(\theta))_{n \geq 0}$: the study of quenched limit theorems for adapted martingales will therefore play an essential role in some of the proofs of the forthcoming results.

5.5 Estimating the Spectral Density

According to Theorem 3.2 in Section 3.2, for a stationary square-integrable process $(X_k)_{k \in \mathbb{Z}} = (T^k X_0)_{k \in \mathbb{Z}}$ (Definition 4.1), the averaged discrete Fourier transforms $S_n(\theta)/n$ (see Definition 2.6) converge almost surely and in $L^2_{\mathbb{P}}$ to a function $P_\theta X_0$ with the property that

$$T P_\theta X_0 = e^{-i\theta} P_\theta X_0.$$

In particular, as we stated in Corollary 3.3, $P_\theta X_0 = 0$ when $e^{-i\theta} \notin \text{Spec}_p(T)$ (Definition 1.3). Even more is true: when \mathcal{F} is countably generated, Proposition 1.4 implies that the set

$$\{\theta \in [0, 2\pi) : P_\theta X_0 \neq 0\} \quad (1.74)$$

has λ -measure zero.

We will go a bit further now by showing that, if $(X_k)_{k \in \mathbb{Z}}$ is a centered process and admits a spectral density then it is given by the asymptotic variance of the properly normalized discrete Fourier transforms. The result is the following:

Theorem 5.4 (Spectral Density as an Asymptotic Variance). *Let $(X_k)_{k \in \mathbb{Z}}$ be a stationary process (Definition 4.1) with $E[X_0] = 0$, and assume that $(X_k)_{k \in \mathbb{Z}}$ admits a spectral density $\sigma^2 : [0, 2\pi) \rightarrow [0, +\infty)$ (Definition 5.3). Then*

$$\sigma^2(\theta) = \lim_n \frac{1}{n} E[|S_n(\theta)|^2] \quad (1.75)$$

in the sense of Definition 2.2 (this is, (1.75) holds for λ -a.e θ), where $S_n(\theta)$ is the n -th discrete Fourier transform of $(X_k)_{k \in \mathbb{Z}}$ at θ (Definition 2.6).

Proof: Let us start with the following observation: given a sequence of complex numbers $(c_k)_{k \in \mathbb{Z}}$, it is not hard to see that

$$\sum_{j=0}^{n-1} \sum_{k=0}^{n-1} c_{j-k} = \sum_{j=-(n-1)}^{n-1} (n - |j|) c_j = \sum_{j=0}^{n-1} \sum_{k=-j}^j c_k. \quad (1.76)$$

Now assume that $(X_k)_{k \in \mathbb{Z}} = (T^k X_0)_{k \in \mathbb{Z}}$ is a stationary process with $E[X_0] = 0$. Denoting by $\gamma : \mathbb{Z} \rightarrow \mathbb{C}$ the autocovariance function of $(X_k)_{k \in \mathbb{Z}}$ ($\gamma(k) = E[X_0 \overline{X_k}]$) and taking

$$c_j := E[X_0 \overline{X_{-j}}] e^{ij\theta} = \gamma(-j) e^{ij\theta}$$

it follows from (1.76) that

$$\begin{aligned} E \left| \frac{1}{\sqrt{n}} S_n(\theta) \right|^2 &= \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} E[X_j \overline{X_k}] e^{ij\theta} e^{-ik\theta} = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} E[X_0 \overline{X_{-(j-k)}}] e^{i(j-k)\theta} = \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=-j}^j E[X_0 \overline{X_{-k}}] e^{ik\theta}. \end{aligned}$$

and the conclusion follows at once from Definition 5.3 and Theorem 5.2. \square

Conclusive Remarks

As we have suggested along the previous sections, this monograph is devoted to the (quenched) asymptotic behavior of the normalized averages

$$A_n(\theta, \omega) = \frac{1}{n} \sum_{k=0}^{n-1} X_k(\omega) e^{ik\theta} = \frac{1}{n} S_n(\theta) \quad (1.77)$$

of the discrete Fourier transforms of a stationary, square-integrable centered process $(X_k)_{k \in \mathbb{Z}} = (T^k X_0)_{k \in \mathbb{Z}}$ (Definition 4.1) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We have seen (Corollary 3.3, Proposition 1.4, and Theorem 5.4) that when \mathcal{F} is countably generated (Definition 1.5) and $(X_k)_{k \in \mathbb{Z}}$ admits a spectral density $\sigma^2 : [0, 2\pi) \rightarrow [0, +\infty)$ (Definition 5.3), there exists a set $I \subset [0, 2\pi)$ of λ -measure one such that for every $\theta \in I$,

$$A_n(\theta, \cdot) \rightarrow_{n \rightarrow \infty} 0, \quad \mathbb{P}\text{-a.s. and in } L^2_{\mathbb{P}}.$$

and

$$E[|\sqrt{n}A_n(\theta, \cdot)|^2] \rightarrow_n \sigma^2(\theta).$$

The next obvious task in this direction is to explore the validity of the central limit theorem for the normalized averages $(\sqrt{n}A_n(\theta, \cdot))_{n \geq 0}$. This is, we would like to have an answer to (for instance) the following questions: if $N(0, \Sigma)$ denotes a centered normal (2-dimensional) random variable with covariance matrix Σ ,

Question 1: *Can we choose the set I with the property that for every $\theta \in I$*

$$\sqrt{n}A_n(\theta, \cdot) \Rightarrow_{n \rightarrow \infty} N(0, \Sigma(\theta)) \quad (1.78)$$

and if so...

Question 2: *Can we actually prove the functional form of this convergence? This is, can we prove that I can be chosen such that the sequence of random functions $(B_n(\theta, \cdot))_{n \geq 0}$ defined on $[0, +\infty)$ by*

$$B_n(\theta, \omega)(t) = \frac{S_{\lfloor nt \rfloor}(\theta)}{\sqrt{n}}(\omega) \quad (1.79)$$

converge weakly (in a sense to be specified later) to a 2-dimensional Brownian motion $B(\theta)$?

Peligrad and Wu, in [41], answered the first question positively for real-valued processes under the regularity condition (1.58). Their result is the following:

Theorem 5.5 (CLT for Discrete Fourier Transforms). *If $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ is a T -filtration and $(X_k)_{k \in \mathbb{Z}} = (T^k X_0)_{k \in \mathbb{Z}}$ is a stationary real-valued, $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ -adapted, and regular process (Definitions 4.1, 4.2 and 5.4) with spectral density $\theta \mapsto \sigma^2(\theta)$ (Definition 5.3), there exists $I \subset [0, 2\pi)$ with $\lambda(I) = 1$ such that for every $\theta \in I$, (1.78) holds, where*

$$\Sigma(\theta) = \begin{bmatrix} \sigma^2(\theta)/2 & 0 \\ 0 & \sigma^2(\theta)/2 \end{bmatrix}. \quad (1.80)$$

Equivalently, for every $\theta \in I$,

$$\sqrt{n}A_n(\theta, \cdot) \Rightarrow_n (\sigma^2(\theta)/2)^{1/2}(N_1 + iN_2),$$

where N_1, N_2 are standard (real valued) independent normal random variables.

With regards to the second question, Peligrad and Wu were able to provide an “intermediate” answer by considering the random functions $B(\theta, \cdot)$ as random elements defined on the probability space

$$([0, 2\pi) \times \Omega, \mathcal{B} \otimes \mathcal{F}, \lambda \times \mathbb{P})$$

To present their result, recall first that a two dimensional or *complex* standard Brownian motion is a random function of the form $B_1 + iB_2$ where B_1, B_2 are independent standard Brownian motions. Peligrad and Wu’s invariance principle is the following.

Theorem 5.6 (FCLT (averaged frequency) for Discrete Fourier Transforms). *Consider $D([0, 1], \mathbb{C})$, the space of cadlag complex-valued functions $[0, 1) \rightarrow \mathbb{C}$ endowed with the Skorohod topology (see Section 7) and, in the context of Theorem 5.5, let $B = B_1 + iB_2$ be a two-dimensional standard Brownian motion defined on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. Then the random functions*

$$V_n : ([0, 2\pi) \times \Omega, \mathcal{B} \otimes \mathcal{F}, \lambda \times \mathbb{P}) \rightarrow D([0, 1], \mathbb{C})$$

specified by (1.79) converge weakly (i.e. in distribution) to the random function $([0, 2\pi) \times \Omega', \mathcal{B} \otimes \mathcal{F}', \lambda \times \mathbb{P}') \rightarrow D([0, 1], \mathbb{C})$ defined by

$$B(\theta, \omega') := \frac{\sigma(\theta)}{\sqrt{2}} B(\omega').$$

The main purpose of this work is to extend these theorems to similar results about *quenched* convergence, a notion that will be explained and explored in Chapter 3. It is important to mention that the results that we will obtain can be considered as generalized versions -under the respective hypotheses- of Theorems 5.5 and 5.6, in the sense that quenched convergence is, as we shall see, a notion of convergence stronger than convergence in distribution (we will also be able to deduce the convergence in Theorem 5.5 from our quenched central limit theorem for Fourier transforms, Theorem 15.1). We will also see that our quenched results are strictly stronger than the ones given here: while it is not difficult to show in a general sense that quenched convergence is strictly stronger than convergence in distribution (see Example 3 in page 59), for the processes under consideration a more careful analysis will be needed (see Theorem 16.3 and the comments following it).

Chapter 2

Convergence in Distribution

In this chapter we discuss several topics related to the notion of convergence in distribution of random elements in a metric space S , specialized to the cases that we will address in subsequent chapters. This will be important in order to understand the methods involved the proofs of the results presented in Chapter 4.

First we will prove, in Section 6, a further equivalence to the Portmanteau theorem (Theorem 6.1), valid in the case in which the underlying metric space S is separable. The important point of this reduction is that the family of functions to be tested for verifying convergence in distribution can be reduced in this case to a countable one, a fact that we will use to prove that our notion of quenched convergence (to be given in Chapter 3) specializes “in the right way” to the case under our consideration (this is, to the setting of regular conditional expectations).

Then, in Section 7, we will discuss briefly the notion of convergence in $D[[0, \infty), \mathbb{C}]$, in order to set the ground for the proofs of the forthcoming results involving the convergence in distribution of complex valued *cadlag* functions. The discussion will give rise to Theorem 7.1, which will be the key to proceed when addressing the proofs of the invariance principles in Chapter 4.

We will then present, in Section 8, some results about convergence of types, which will be used on our discussions relating the possible “quenched” and “annealed” limits of a stochastic process, a discussion that will be essential to prove that the annealed limit theorems of Fourier transforms inspiring our results cannot themselves be extended to quenched ones: a random normalization is necessary (see Theorem 16.3, and the discussion preceding its statement, in page 80).

In the section devoted to “random elements and product spaces” (Section 9) we will address the relationship within the convergence in distribution of a sequence of random elements $(Z_k)_k$ depending on two (random) parameters (θ, ω) for a.e. fixed θ and the convergence in distribution of this sequence on the product space of the domain of the parameters. This discussion will serve later to clarify the hierarchy between the invariance principles under our consideration.

Finally, in Section 10, we will present a result (Theorem 10.1) used along our arguments in order to transport the asymptotic distributions of the processes under our consideration from suitable martingale approximations.

6 A Refinement of the Portmanteau Theorem

Throughout this section (S, \mathcal{T}) will denote a topological space with topology \mathcal{T} . If S is a metric space, we will use the notation (S, d) , where $d : S \times S \rightarrow [0, +\infty)$ is the corresponding metric.

To begin with, remember the notion of a *Urysohn function*.

Definition 6.1 (Urysohn Function). *Given two closed, disjoint sets F_0, F_1 in a perfectly normal topological space (for instance, any metric space) (S, \mathcal{T}) , a function*

$$U(F_0, F_1) : S \rightarrow [0, 1] \quad (2.1)$$

*is called a **Urysohn function** if it is continuous, $U^{-1}\{0\} = F_0$ and $U^{-1}\{1\} = F_1$.*

Remark 6.1. The existence of Urysohn functions for every two disjoint closed sets is the axiom characterizing perfectly normal spaces, and it is a standard fact from general topology that metrizable spaces are perfectly normal. We also recall the following: *for a perfectly normal space every closed set F is a G_δ -set*: there exists a countable family $\{G_k\}_{k \in \mathbb{N}}$ of open sets such that $F = \bigcap_{k \in \mathbb{N}} G_k$.

The following definition is introduced for technical purposes.

Definition 6.2 (Co-base). *Given a topological space (S, \mathcal{T}) , let us call a collection $\{F_j\}_{j \in J}$ of closed subsets of S a **co-base** if $\{S \setminus F_j\}_{j \in J}$ is a base of \mathcal{T} .*

Note that if (S, d) is separable it admits a co-base that is also a π -system (consider the finite intersections on any co-base). In the sequel, $\mathbf{C}^b(S)$ **denotes the space of continuous and bounded functions** $f : S \rightarrow \mathbb{R}$. If needed, we will consider it also as a metric space via the *uniform norm*

$$\|f\|_\infty := \sup_{s \in S} |f(s)|, \quad (2.2)$$

for every $f \in \mathbf{C}^b(S)$.

Theorem 6.1 (A Refinement of the Portmanteau Theorem). *Let S be a separable metric space, let $\{F_n\}_{n \in \mathbb{N}}$ be a co-base of S which is also a π -system, and let X_n, X ($n \in \mathbb{N}$) be random elements of S (Definition 2.3)¹. Then the following two statements are equivalent*

1. *For every $f \in \mathbf{C}^b(S)$,*

$$\lim_n E f(X_n) = E f(X).$$

¹Note that the X_n 's are not necessarily defined on the same probability space.

2. For every $k \in \mathbb{N}$, every rational $\epsilon > 0$, and some Urysohn function $U_{k,\epsilon} = U(S \setminus F_k^\epsilon, F_k)$

$$\lim_n EU_{k,\epsilon}(X_n) = EU_{k,\epsilon}(X),$$

where F_k^ϵ is given according to Definition 2.1.

As stated before, the importance of this theorem for our purposes resides in the fact that it allows us to reduce the family of test functions in Portmanteau's Theorem to a countable one, a fact that will be exploited in the proof of Proposition 13.1 in Chapter 3.

Proof of Theorem 6.1: Denote by P_n the law of X_n and by P the law of X (see Definition 2.3). Since 1. clearly implies 2. it suffices to see, by the Portmanteau Theorem ([10], Theorem 2.1), that if 2. is true then for any given closed set F

$$\limsup_n P_n F \leq P F.$$

If for some k , $F = F_k$, this is a consequence of the inequality

$$I_F \leq U_{k,\epsilon} \leq I_{F^\epsilon},$$

the hypothesis in 2. and the continuity from above of finite measures.

If F is an arbitrary closed set, say $F = \bigcap_{j \in J} F_j$ for some $J \subset \mathbb{N}$, and if we define for all $k \in \mathbb{N}$, $J_k := J \cap [0, k]$ and $A_k := \bigcap_{j \in J_k} F_j$ then, since $A_k \in \{F_n\}_n$,

$$\limsup_n P_n F \leq \limsup_n P_n A_k \leq P A_k$$

for all k . By letting $k \rightarrow \infty$ we get the desired conclusion. \square

Remark 6.2. We remark that the Portmanteau theorem can be extended to the context of random elements in abstract perfectly normal spaces (with their Borel sigma algebra) if one interprets “convergence in distribution” as the fulfillment of the hypothesis 1. of Theorem 6.1. This can be seen by following the arguments in [10] and using the fact that every closed set is a G_δ set (Remark 6.1). In this context, Theorem 6.1 corresponds to the second-countable case.

7 Convergence of Complex-valued Cadlag Functions

This monograph contains results about convergence in distribution (under several measures) of random elements of $D[[0, \infty), \mathbb{C}]$: the space of functions $f : [0, \infty) \rightarrow \mathbb{C}$ that are continuous from the right and have left-hand limits at every point (**cadlag** functions in \mathbb{C}). This space is an algebra with the operation of multiplication and addition given by the usual pointwise operations between complex functions, and it is a (\mathbb{C} or \mathbb{R})-vector space with the usual operation of multiplication by constants regarded as constant functions.

To clarify the notions behind our results about convergence in $D[[0, \infty), \mathbb{C}]$, let us start in the following way: first, **denote by $(D[[0, \infty)], d)$ the space of real-valued cadlag functions** endowed with the Skorohod distance d defined in [10], (16.4), which we proceed to describe now for the sake of completeness.

Definition of the Skorohod Distance (*real-valued case*)

Fix $m \in \mathbb{N}^*$, and consider the following definitions

1. First, define the family

$$\Lambda_m := \{\varphi : [0, m] \rightarrow [0, m] : \varphi \text{ is surjective, nondecreasing, and } \|\varphi\|_m < \infty\}, \quad (2.3)$$

where

$$\|\varphi\|_m = \sup_{0 \leq s < t \leq m} \left| \log \frac{\phi(t) - \phi(s)}{t - s} \right|.$$

Note in particular that for every $\varphi \in \Lambda_m$, $\varphi(0) = 0$, $\varphi(m) = m$, and φ is continuous.

2. Now consider the *Skorohod distance* d_m in the space $D([0, m])$ of real-valued *cadlag* functions with domain $[0, m]$: for every $w_1, w_2 \in D([0, m])$

$$d_m(w_1, w_2) = \inf_{\phi \in \Lambda_m} \{ \|\phi\|_m \vee \|w_1 - w_2 \circ \phi\| \} \quad (2.4)$$

where $\|\cdot\|$ denotes the corresponding uniform norm in $D([0, m])$:

$$\|w\| = \sup_{0 \leq t \leq m} |w(t)|.$$

3. Finally denote by r_m the *restriction operator* $r_m : D([0, \infty)) \rightarrow D([0, m])$ given by

$$(r_m w)(t) = w(t), \quad (2.5)$$

define $g_m : [0, \infty) \rightarrow [0, 1]$ by

$$g_m(t) = \begin{cases} 1 & , 0 \leq t \leq m-1 \\ m-t & , m-1 < t \leq m \\ 0 & , m < t \end{cases} \quad (2.6)$$

and denote, for every $w \in D([0, \infty))$

$$w^k := r_k(g_k w). \quad (2.7)$$

With these notations we define d as follows: given $w_1, w_2 \in D([0, \infty))$

$$d(w_1, w_2) := \sum_{k \geq 1} 2^{-k} (1 \wedge d_k(w_1^k, w_2^k)). \quad (2.8)$$

7.1 The topology of $D([0, \infty), \mathbb{C})$

Let $S = D([0, \infty), \mathbb{C})$. The bijection $D([0, \infty), \mathbb{C}) \rightarrow D[0, \infty) \times D[0, \infty)$ given by

$$w = \operatorname{Re}(w) + i\operatorname{Im}(w) \mapsto (\operatorname{Re}(w), \operatorname{Im}(w))$$

allows us to regard S as a topological space whose topology is the topology generated by the product Skorohod topology of $(D[[0, \infty)], d)$, this is, by the product of the topologies induced by the metric (2.8). This topology is metrizable: it is induced by the *product Skorohod metric* denoted (also) by $d : S \times S \rightarrow S$ and given by

$$d(w_1, w_2) := ((d(\operatorname{Re}(w_1), \operatorname{Re}(w_2)))^2 + (d(\operatorname{Im}(w_1), \operatorname{Im}(w_2)))^2)^{1/2} \quad (2.9)$$

where “ d ”, at the right-hand side, is given by (2.8).

Definition 7.1 (The space $D[[0, \infty), \mathbb{C}]$). *The Skorohod distance in $S = D[[0, \infty), \mathbb{C}]$ is the distance d defined by (2.9). (S, d) is the space of **cadlag complex-valued functions on $[0, \infty)$** .*

Remark 7.1 (A Criterion for Measurability). Let $\mathcal{D}_{\infty, \mathbb{C}}$ be the Borel sigma algebra on $S = D[[0, \infty), \mathbb{C}]$, let (Ω, \mathcal{F}) be a measurable space, and let $X : \Omega \rightarrow S$ be a given function.

By our definition of the topology of S , to prove that X is $\mathcal{F}/\mathcal{D}_{\infty, \mathbb{C}}$ measurable it suffices to see the measurability of the real and imaginary parts of X . This observation, combined with the argument in [10], p.84, and with Theorem 16.6 in that book shows that X is $\mathcal{F}/\mathcal{D}_{\infty, \mathbb{C}}$ -measurable if and only there exists a dense set $T \subset [0, \infty)$ such that for every $t \in T$, $\omega \mapsto X(\omega)(t)$ is \mathcal{F} -measurable.

Finally let us point out that, since separability and completeness ascend to the product space (with the product metric), and $(D[[0, \infty)], d)$ is separable and complete ([10], Theorem 16.3), we have the following proposition.

Proposition 7.1. *The space of cadlag complex-valued functions on $[0, \infty)$ (Definition 7.1) is separable and complete.*

7.2 Convergence on $D[[0, \infty), \mathbb{C}]$

The space of cadlag complex-valued function admits, as any other metric space, a notion of convergence in distribution. To prove that an actual sequence of random elements in this space converges in distribution we will use the theoretical framework explained in [10] for convergence of real-valued cadlag functions, whose arguments can be transported to the case of complex-valued functions without major difficulties. More precisely, we will prove convergence in $D[[0, \infty), \mathbb{C}]$ via the following facts:

1. *Generic Idea.* Let us start by recalling the generic idea: remember that a sequence $(P_n)_{n \in \mathbb{N}}$ of probability measures on a metric space is *tight* if for any $\epsilon > 0$ there exists a compact set K such that for every $n \in \mathbb{N}$, $P_n(K) > 1 - \epsilon$, and that when the space is separable and complete, tightness is equivalent to the *relative compactness* of $(P_n)_{n \in \mathbb{N}}$ (see [10], Theorems 5.1 and 5.2): $(P_n)_{n \in \mathbb{N}}$ is tight if and only if for every (strictly increasing) sequence $(n_k)_{k \in \mathbb{N}}$ of natural numbers there exists a subsequence $(n_{k'})_{k'}$ and a probability measure P' with $P_{n_{k'}} \Rightarrow P'$. It follows that a tight sequence is convergent if P' is independent of the given (sub)sequences. Since tightness is a necessary condition for convergence of measures, a way of addressing proofs of convergence in distribution is to give criteria for tightness and criteria to identify asymptotic distributions so that, in practice, one proves that a sequence of

probability measures is convergent by proving that it is tight and that there exists a unique subsequential distribution via these criteria.

2. *Criteria for Tightness.* Now, if $W : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow D[[0, \infty), \mathbb{C}]$ is a random element of $D[[0, \infty), \mathbb{C}]$, then the inequalities

$$\mathbb{P}[W \notin K_1 \times K_2] \leq \mathbb{P}[\operatorname{Re}(W) \notin K_1] + \mathbb{P}[\operatorname{Im}(W) \notin K_2] \leq 2\mathbb{P}[W \notin K_1 \times K_2]$$

show that, given a sequence $(W_n)_{n \in \mathbb{N}}$ of random elements in $D[[0, \infty), \mathbb{C}]$, $(\operatorname{Re}(W_n))_n$ and $(\operatorname{Im}(W_n))_n$ are tight if and only if $(W_n)_n$ is tight. Of course, this argument shows (the well known fact) that a sequence of random elements in the product of two metric spaces is tight if and only if the component sequences are tight.

The important observation is that we can prove tightness in $D[[0, \infty), \mathbb{C}]$ by applying criteria for tightness in $D[[0, \infty)]$ to the real and imaginary parts of any given random sequence of cadlag complex-valued functions.

3. *Asymptotic Distributions.* By an adaptation of the arguments in [9] and [10], it is possible to show that the finite dimensional distributions are a *separating class* in $D[[0, \infty), \mathbb{C}]$: if for every $t_1 \leq \dots \leq t_n$ we denote by $\pi_{t_1 \dots t_k} : D[[0, \infty)] \rightarrow \mathbb{C}^k$ the projection

$$\pi_{t_1 \dots t_k}(w) := (w(t_1), \dots, w(t_k)), \quad (2.10)$$

then two measures P_1 and P_2 in $D[[0, \infty), \mathbb{C}]$ coincide if and only if there exists a dense subset $T \subset [0, \infty)$ such that for every $0 \leq t_1 \leq \dots \leq t_n$ in T the n -th dimensional distributions $P_j \pi_{t_1 \dots t_k}^{-1}$ ($j = 1, 2$) on $\mathbb{C}^n = \mathbb{R}^{2n}$ are the same.

4. *Restriction of the domains.* Let P be a probability measure in $D[[0, \infty), \mathbb{C}]$ and, for $m > 0$, consider the space $D([0, m], \mathbb{C})$ of cadlag complex-valued functions on $[0, m]$ with the Skorohod distance defined by identifying $D([0, m], \mathbb{C}) = D([0, m]) \times D([0, m])$ and extending d_m (see (2.4)) to the product space as above. If m is such that

$$P\{w : \lim_{t \rightarrow m^-} w(t) \neq w(m)\} = 0 \quad (2.11)$$

and $r_m : D[[0, \infty), \mathbb{C}] \rightarrow D([0, m], \mathbb{C})$ is the restriction operator defined above ($(r_m w)(t) = w(t)$), then the hypothesis $P_n \Rightarrow P$ in $D[[0, \infty), \mathbb{C}]$ implies that $\mathbb{P}_n r_m^{-1} \Rightarrow \mathbb{P} r_m^{-1}$ in $D([0, m], \mathbb{C})$, and the following “converse” holds: if $(m_k)_k$ is a sequence increasing to infinity such that (2.11) holds for all $m = m_k$, then the hypothesis

$$P_n r_{m_k}^{-1} \Rightarrow P r_{m_k}^{-1} \quad \text{for every } k \in \mathbb{N}$$

(on $D([0, m_k], \mathbb{C})$) implies that $P_n \Rightarrow P$. This observation allows us to prove convergence in $D[[0, \infty), \mathbb{C}]$ by restricting our attention to $D([0, m], \mathbb{C})$.

Let us finish this section by giving a more concrete criterion for convergence in $D[[0, \infty), \mathbb{C}]$. The proof will be just briefly sketched using the facts recalled here and referring to additional arguments from [9] and [10].

Theorem 7.1 (Criterion of Convergence). *Let P_n, P be probability measures in $D[[0, \infty), \mathbb{C}]$ ($n \in \mathbb{N}$) and consider, for every $t > 0$, the set*

$$J_t = \{w \in D[[0, \infty), \mathbb{C}] : \lim_{s \rightarrow t^-} w(s) \neq w(t)\}.$$

Then the set A_P of nonnegative numbers t such that $PJ_t = 0$ has a countable complement (in $[0, \infty)$), and if $(P_n)_{n \in \mathbb{N}}$ is tight, $P_n \Rightarrow P$ if and only if for every $t_1 \leq \dots \leq t_k$ in A_P

$$P_n \pi_{t_1 \dots t_k}^{-1} \Rightarrow_n P \pi_{t_1 \dots t_k}^{-1}. \quad (2.12)$$

Proof (sketch): To see that A_P has a countable complement show, using the argument in [9], p.124 that for every given $m \geq 0$, $[0, m] \setminus T_P$ is countable.

To prove the second statement assume that $(P_n)_{n \in \mathbb{N}}$ is tight, and start by considering a strictly increasing sequence $(s_k)_{k \in \mathbb{N}}$ of elements in A_P with $\lim_k s_k = \infty$.

Given $m > 0$, denote by r_m the restriction operator given by (2.5). By 4. above, $P_n \Rightarrow P$ if and only if given $m \in \{s_k\}_{k \in \mathbb{N}}$, $P_n r_m^{-1} \Rightarrow P r_m^{-1}$ as $n \rightarrow \infty$.

Now, by an adaptation of Theorem 15.1 in [9]: $P_n r_m^{-1} \Rightarrow P r_m^{-1}$ if and only if $(P_n r_m^{-1})_{n \in \mathbb{N}}$ is tight and for every $0 \leq t_1 \leq \dots \leq t_k \leq m$

$$P_n (\pi_{t_1 \dots t_k}^m r_m)^{-1} \Rightarrow P (\pi_{t_1 \dots t_k}^m r_m)^{-1}, \quad (2.13)$$

where $\pi_{t_1 \dots t_k}^m : D([0, m], \mathbb{C}) \rightarrow \mathbb{C}^k$ denotes the projection specified by (2.10) (the superindex “ m ” is introduced to indicate the domain).

The tightness of $(P_n r_m^{-1})_{n \in \mathbb{N}}$ follows by an adaptation of the argument at the beginning of the proof of Theorem 16.7 in [10] (in short: r_m is continuous because $m \in A_P$, and since $(P_n)_{n \in \mathbb{N}}$ is tight, $(P_n r_m^{-1})_{n \in \mathbb{N}}$ is tight by the mapping theorem), and therefore it suffices to prove, by Theorem 15.1 in [9] again, that the convergence (2.12) for every $0 \leq t_1 \leq \dots \leq t_k$ is equivalent to the convergence (2.13) for every $m \in A_P$ and every $0 \leq t_1 \leq \dots \leq t_k \leq m$, but this is just a consequence of the equality

$$\pi_{t_1 \dots t_k}^m r_m = \pi_{t_1 \dots t_k}$$

(where $m \geq t_k$) and the fact that $\lim_n s_n = \infty$. □

8 Convergence of Types

In this section we present some facts about Convergence of Types in a form that is convenient for the proofs given along this monograph. Let us start by recalling the notion of a *non-degenerate* distribution function.

Definition 8.1 (Nondegenerate Distribution Function). *A distribution function F is **non-degenerate** if it is not the indicator function of some interval $[a, +\infty)$. This is, if it does not correspond to a constant random variable.*

Our arguments in this section will be mostly based on the following *Convergence of Types* theorem ([11], Theorem 14.2). In accordance with the notation introduced at the beginning, if F_n and F are (probability) distribution functions, “ $F_n \Rightarrow_n F$ ” will denote pointwise convergence at the continuity points of F (convergence of distribution functions).

Lemma 2 (Convergence of Types). *Let F_n , F and G be distribution functions, and let a_n, u_n, b_n, v_n be constants with $a_n > 0$, $u_n > 0$. If F , G are non-degenerate, $F_n(a_n x + b_n) \Rightarrow F(x)$, and $F_n(u_n x + v_n) \Rightarrow G(x)$ then there exist $a = \lim_n a_n/u_n$, $b = \lim_n (b_n - v_n)/u_n$ and $G(x) = F(ax + b)$.*

Note that, necessarily, $a > 0$ (as otherwise G would be constant).

We will translate this statement to a statement about convergence of stochastic processes (with a restricted choice of u_n, v_n see Proposition 3 below), which we will be able to extend to the complex valued case.

8.1 Preliminary Facts

To begin with, we remind the following elementary facts, here capital letters (U_n , V_n , etc) denote (real-valued) random variables and “ \rightarrow_P ” denotes convergence in probability. For more details see for instance Section 25 in [11].

1. If a is constant then $U_n \Rightarrow a$ if and only if $U_n \rightarrow_P a$.
2. If $U_n \Rightarrow W$ and $V_n \rightarrow_P 0$ then $U_n + V_n \Rightarrow W$.
3. If $(a_n)_n$ is a sequence of constant functions then $a_n \Rightarrow A$ if and only if $a = \lim_n a_n$ exists (and therefore $A = a$ a.s.).

These facts will be used along the proof of the forthcoming results in this section, which will be useful when addressing the issues of the relationship between “annealed” convergence and “quenched” convergence of a sequence of random variables.

8.2 Convergence of Types Results

In this section all the processes under consideration will be assumed real-valued, unless otherwise specified. We will simply write “ $V_n \Rightarrow V$ ” to indicate that the stochastic process $(V_n)_{n \in \mathbb{N}}$ converges in distribution to V as $n \rightarrow \infty$.

Lemma 3. *If $Y_k \Rightarrow Y$ and $\{c_k\}_k \subset \mathbb{R}$ are such that $Y_k + c_k \Rightarrow 0$, then $Y = -\lim_k c_k$. In particular, Y is a constant function.*

Proof: Note that $c_k = -Y_k + (Y_k + c_k) \Rightarrow -Y$ because $Y_k + c_k \Rightarrow 0$. Now use 3. in Section 8.1. \square

Corollary 8.1. *If X is not constant, $X_n \Rightarrow X$, and a_n, b_n are such that $a_n X_n + b_n \Rightarrow 0$, then $a_n \rightarrow 0$ and $b_n \rightarrow 0$.*

Proof: If $0 < a := \limsup_n a_n \leq \infty$ and $a_{n_k} \rightarrow_{k \rightarrow \infty} a$ with $a_{n_k} > 0$, then applying Lemma 3 with $Y_k = X_{n_k}$ and $c_k = b_{n_k}/a_{n_k}$ we conclude that X is constant. This proves that, necessarily, $\limsup_n a_n \leq 0$. A similar argument shows that $\liminf_n a_n \geq 0$, and therefore $\lim_n a_n = 0$.

The fact that $b_n \rightarrow 0$ follows from here applying Lemma 3 again, because $a_n X_n \Rightarrow 0$. \square

These results give rise to the following proposition

Proposition 8.1. *If X is not constant, $X_n \Rightarrow X$ and $a_n > 0$, b_n are such that $a_n X_n + b_n \Rightarrow Y$, then there exists $a = \lim_n a_n$, $b = \lim_n b_n$ and, therefore, $Y = aX + b$ (in distribution).*

Proof: If Y is constant then, from $a_n X_n + b_n - Y \Rightarrow 0$ (see 1. in Section 8.1) it follows, via Corollary 8.1, that $\lim_n a_n = 0$ and $\lim_n b_n = Y$.

If Y is not constant we apply Lemma 2 with F_n , F , and G the distribution functions of X_n , X and Y respectively, and with $u_n = 1$, $v_n = 0$. \square

Remark 8.1. Taking $X_n = 1$ (the constant function), $a_n = n$, and $b_n = -n$, we see that the given restriction on X (to be non constant) is necessary. The asymptotically degenerate case is nonetheless covered by the following proposition (note the additional restriction on the coefficient of X_k).

Proposition 8.2. *If X is constant, $X_k \Rightarrow X$, and $X_k + c_k \Rightarrow Z$ then $c = \lim_k c_k$ exists and therefore $Z = X + c$ (in distribution).*

Proof: Use $X + c_k = (X - X_k) + X_k + c_k \Rightarrow Z$ by 2. in Section 8.1. The conclusion follows from the item 3. there.

These results can be transported to the case of complex-valued random elements. More concretely.

Proposition 8.3 (Convergence of Types for complex-valued Random Variables). *Proposition 8.1 and Proposition 8.2 remain valid if the processes involved are complex-valued, provided that the constants $(a_n)_n$ in Proposition 8.1 are still real and positive (all the other constants can be assumed complex).*

Proof: To see this for Proposition 8.1 notice that if X_n, X are complex valued, $X_n \Rightarrow X$, and $\mathbf{u} \in \mathbb{C} = \mathbb{R}^2$ is any vector then, by the mapping theorem

$$\mathbf{u} \cdot (a_n X_n + b_n) = a_n (\mathbf{u} \cdot X_n) + \mathbf{u} \cdot b_n \Rightarrow \mathbf{u} \cdot Y$$

so that, by the real valued case just proved, there exists $a = \lim_n a_n$ and $b_{\mathbf{u}} = \lim_n \mathbf{u} \cdot b_n$. Since \mathbf{u} is arbitrary, there actually exists $b = \lim_n b_n$.

The second conclusion ($Y = aX + b$ in distribution) follows at once from the Cramer-Wold theorem. The argument for Proposition 8.2 is similar. \square

9 Random Elements and Product Spaces

Since we will be concerned with random cadlag functions seen as random elements depending on two random parameters (θ, ω) or on a single parameter ω for θ fixed (see Chapter 4), it is convenient to give now the following proposition.

Proposition 9.1 (Convergence for fixed parameters and on the Product Space). *Let (S, d) be a metric space and let $(\Theta, \mathcal{B}, \lambda)$ be a probability space.² Assume that for every $\theta \in \Theta$ and every $n \in \mathbb{N} \cup \{\infty\}$, $V_n(\theta)$ is a random element (Definition 2.3) in S defined on a probability space $(\Omega_n, \mathcal{G}_n, \mathbb{P}_n)$, and that the function V_n given by $(\theta, \omega) \mapsto V_n(\theta)(\omega)$ is measurable with respect to the product sigma-algebra $\mathcal{B} \otimes \mathcal{G}_n$. If for λ -a.e θ $V_n(\theta) \Rightarrow V_\infty(\theta)$ as $n \rightarrow \infty$, then the random elements $V_n : (\Theta \times \Omega_n, \mathcal{B} \otimes \mathcal{G}_n, \lambda \times \mathbb{P}_n) \rightarrow S$ converge in distribution to the random element $V_\infty : (\Theta \times \Omega_\infty, \mathcal{B} \otimes \mathcal{G}_\infty, \lambda \times \mathbb{P}_\infty) \rightarrow S$ as $n \rightarrow \infty$.*

Proof: Given any bounded and continuous function $f : S \rightarrow \mathbb{R}$, consider the function

$$g_n(\theta) := E(f(V_n(\theta))) - E f(V(\theta))$$

where (we emphasize again that) “ E ” denotes integration with respect to the respective probability measures $(\mathbb{P}_n$ and $\mathbb{P}_\infty)$. By the hypotheses on $V_n(\theta)$ and Fubini’s theorem, g_n is \mathcal{B} -measurable, and since $V_n(\theta) \Rightarrow V_\infty(\theta)$ for λ -a.e θ , $g_n(\theta) \rightarrow 0$ as $n \rightarrow \infty$, λ -a.s. It follows from the dominated convergence theorem that

$$\int_{\Theta} g_n(\theta) d\lambda(\theta) \rightarrow_n 0$$

as $n \rightarrow \infty$. This is (Fubini’s Theorem again), that

$$\int_{\Theta \times \Omega} f \circ V_n d(\lambda \times \mathbb{P}_n) \rightarrow_n \int_{\Theta \times \Omega} f \circ V_\infty d(\lambda \times \mathbb{P}_\infty),$$

which gives the desired conclusion. \square

The following example shows that the converse of Proposition 9.1 does not hold.

Example 2. Consider the probability space $([0, 2\pi), \mathcal{B}, \lambda)$ and let, for every $n \geq 0$, $f_n : [0, 2\pi) \rightarrow [0, \infty)$ be a sequence of (\mathcal{B} -measurable) functions with the property that $f_n \rightarrow 0$ in L^1_λ and for every $\theta \in [0, 2\pi)$, $(f_n(\theta))_{n \in \mathbb{N}}$ is not convergent. For instance take $f_0 = I_{[0, 2\pi)}$, $f_1 = I_{[0, \pi)}$, $f_2 = I_{[\pi, 2\pi)}$, $f_3 = I_{[0, \pi/2)}$, $f_4 = I_{[\pi/2, \pi)}$, and so on.

Given any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let X be the constant function $X(\omega) = 1$ and consider, for every $n \in \mathbb{N}$, the random variable $V_n : ([0, 2\pi) \times \Omega, \mathcal{B} \otimes \mathcal{F}, \lambda \times \mathbb{P}) \rightarrow \mathbb{R}$ given by $V_n(\theta, \omega) = f_n(\theta)X(\omega) = f_n(\theta)$. Since $f_n \rightarrow 0$ in L^1_λ , $V_n \Rightarrow 0$, but note that since the law of X is the Dirac measure δ_1 ($\delta_1\{1\} = 1$), the sequence of random variables $(V_n(\theta, \cdot))_{n \in \mathbb{N}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ does not converge in distribution for any $\theta \in [0, 2\pi)$ (the law of $V_n(\theta, \cdot)$ is $\delta_{f_n(\theta)}$).

As this discussion shows, given a sequence $(V_n)_{n \in \mathbb{N}}$ as in the statement of Proposition 9.1, the convergence in distribution of $V_n(\theta, \cdot)$ for λ -a.e θ is in general a notion *stronger* than that of the convergence in distribution of V_n . We will return to this discussion in Chapter 4.

²This is just *some* probability space but, as the notation suggests, we will use only the case $\Theta = [0, 2\pi)$ with the Borel sigma algebra and the normalized Lebesgue measure.

10 A Transport Theorem

The last result to be recalled in this chapter, Theorem 10.1, is an improvement due to Dehling, Durieu and Volný, of Theorem 3.1 in [10] for the case in which the target (state) space is a complete and separable metric space.

Theorem 10.1 (Transport Theorem). *Let (S, d) be a complete and separable metric space. Assume that for all natural numbers r, n , $X_{r,n}$ and X_n are random elements of S defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and that $X_{r,n} \Rightarrow_n Z_r$. Then the hypothesis*

$$\lim_r \limsup_n \mathbb{P}[d(X_{r,n}, X_n) \geq \epsilon] = 0 \quad \text{for all } \epsilon > 0, \quad (2.14)$$

implies the existence of a random element X of S such that $Z_r \Rightarrow_r X$ and $X_n \Rightarrow_n X$.

Proof: This is Theorem 2 in [23]. □

Corollary 10.2. *In the context of Theorem 10.1 denote, for any given $q > 0$,*

$$\|Z\|_{\mathbb{P},q} := \left(\int_{\Omega} |Z|^q d\mathbb{P}(\omega) \right)^{1/q}.$$

If for some $q > 0$

$$\lim_r \limsup_n \|d(X_{r,n}, X_n)\|_{\mathbb{P},q} = 0$$

and if for all (but finitely many) $r \in \mathbb{N}$ there exists a random element Z_r with $X_{r,n} \Rightarrow_n Z_r$, then there exists a random element X such that $X_n \Rightarrow_n X$ and $Z_r \Rightarrow_r X$.

Proof: Apply Markov's inequality to verify the hypothesis of Theorem 10.1. □

We will use these results to obtain the asymptotic distributions of the processes under our consideration from suitable martingale approximations.

Chapter 3

Quenched Convergence and Regular Conditional Expectations

In this chapter we introduce the notions of *quenched convergence with respect to a sigma algebra* and *regular conditional expectation*. These notions will settle the formal ground for our discussions on asymptotic limit theorems “started at a point”.

Results on quenched convergence -in particular those related to the asymptotics of averages for dependent structures- are the object of intensive research at the moment of writing this monograph (see for instance [4], [5], [18], [21], [46] and the references therein), but they have been in the literature for at least about forty years (see for instance Theorem 20.4 in [9]). These results belong to the category of limit theorems for nonstationary processes: in loose terms, they refer to convergence in distribution of a process with respect to a family of random measures that “integrate” to a stationary distribution for the process in question.¹

The presentation is organized as follows: in Section 11 we introduce the notions of *quenched convergence* of a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a sub sigma-algebra \mathcal{F}_0 of \mathcal{F} , and the notion of *regularity* for conditional expectations, which will be an important assumption along our forthcoming proofs, and we give some basic properties associated to this notions.

Then, in Section 12, we provide some examples of probability spaces and initial sigma algebras that admit a regular conditional expectation. To be more precise, we show (Example 6) that this is the case for the setting of functions of stationary Markov Chains, which encompasses a broad family of the processes present in the applications.

We move then to quickly discuss, in Section 13, the relationship between regularity and quenched convergence. We prove there (Proposition 11.2) that, in the case of a separable state space, quenched convergence with respect to a sigma algebra \mathcal{F}_0 admitting a regular conditional expectation is the same as convergence in distribution with respect to any

¹See Definition 11.2 and Proposition 11.2 for the precise meaning of this statement.

family of probability measures decomposing $E[\cdot | \mathcal{F}_0]$, an assumption that is apparently implicit in the literature but whose proof is not present among the visible references.

Finally, we show that the notion of regular conditional expectation behaves well with respect to the product of probability spaces: the product of two sigma-algebras admitting regular conditional expectations satisfies itself this property, and a decomposition of the expectation with respect to this product sigma-algebra is given by the product of any two decompositions of the factor algebras (Proposition 14.1).

11 Definitions and General Remarks

In this section we introduce the notions of *quenched convergence* of a stochastic process and *regular conditional expectation* with respect to a sigma algebra.

11.1 Quenched Convergence

Let $(Y_n)_{n \geq 1}$ be a measurable sequence on some metric space (S, d) . This is, for every n ,

$$Y_n : (\Omega, \mathcal{F}) \rightarrow (S, \mathcal{S}) \quad (3.1)$$

is an \mathcal{F}/\mathcal{S} measurable function where (Ω, \mathcal{F}) is a (fixed) measure space and \mathcal{S} is the Borel sigma algebra of S . Let \mathbb{P} be a given probability measure on (Ω, \mathcal{F}) , so that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and denote by “ $\Rightarrow_{\mathbb{P}}$ ” the convergence in distribution with respect to \mathbb{P} .

The Portmanteau theorem ([10], Theorem 2.2) states, among other equivalences, that if $Y : (\Omega', \mathcal{F}', \mathbb{P}') \rightarrow (S, \mathcal{S})$ is a random element of S (Definition 2.3) then $Y_n \Rightarrow_{\mathbb{P}} Y$ if and only if for every bounded and continuous function $f : S \rightarrow \mathbb{R}$

$$\int_{\Omega} f \circ Y_n(\omega) d\mathbb{P}(\omega) \rightarrow_{n \rightarrow \infty} \int_{\Omega'} f \circ Y(z) d\mathbb{P}'(z) \quad (3.2)$$

or, in the usual probabilistic notation, if $\lim_{n \rightarrow \infty} Ef(Y_n) = Ef(Y)$, where E is the expectation (Lebesgue integral) with respect to the corresponding probability measures.

A stronger kind of convergence, *quenched convergence*, can be defined in the following way:

Definition 11.1 (Quenched Convergence). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(Y_n)_{n \in \mathbb{Z}}$ be a sequence of random elements on a metric space (S, d) as in (3.1), and let Y be a random element of (S, d) defined on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. Fix a sub-sigma algebra $\mathcal{F}_0 \subset \mathcal{F}$, and denote by E_0 the conditional expectation with respect to \mathcal{F}_0 . We say that Y_n **converges to Y in the quenched sense with respect to \mathcal{F}_0** if for every bounded and continuous function $f : S \rightarrow \mathbb{R}$*

$$E_0[f(Y_n)] \rightarrow_n Ef(Y), \quad \mathbb{P}\text{-a.s.}$$

Remark 11.1. As indicated in Section 4.2, \mathcal{F}_0 will represent in the practice, in a heuristic language, the “initial information” about (or “the past” of) the process $(Y_n)_{n \in \mathbb{Z}}$. In most of our discussions it will be clear from the context what \mathcal{F}_0 is, thus we will simply speak of *quenched convergence* when addressing quenched convergence with respect to \mathcal{F}_0 .

Note also the following: since the convergence in Definition 11.1 is pointwise convergence of uniformly bounded functions (to a constant value), the dominated convergence theorem guarantees that for every continuous and bounded function f

$$\lim_n E[E_0 f(Y_n)] = \lim_n E[f(Y_n)] = Ef(Y),$$

thus, certainly, *quenched convergence implies convergence in distribution*.

Example 3 (Quenched Convergence vs Convergence in Distribution). An example showing that the notion of quenched convergence is strictly stronger than convergence in distribution can be constructed by starting from any sequence $(Y_n)_n$ of \mathcal{F}_0 -measurable functions and noticing that quenched convergence of Y_n to Y in this case is the same as

$$f(Y_n) \rightarrow Ef(Y), \quad \mathbb{P}\text{-a.s.}$$

for all continuous and bounded functions f , which is not possible if, for instance, $(Y_n)_n$ takes the values 1 and 0 infinitely often \mathbb{P} -a.s. Thus it suffices to consider a sequence $(Y_n)_n$ of random variables that converges in distribution but gives \mathbb{P} -a.s a sequence with infinitely many 0's and 1's, and then to define $\mathcal{F}_0 := \sigma(Y_n)_n$: take for instance the functions f_n in Example 2 or, for an even simpler example, consider unit interval with Lebesgue measure as the underlying probability space and, for every $k \in \mathbb{N}$, define $Y_{2k} := I_{[0, 1/2]}$ and $Y_{2k+1} := I_{(1/2, 1]}$.

Now recall the following property of conditional expectations (for a proof see for instance Theorem 34.2 (v) in [11]):

Theorem 11.1 (Dominated Convergence Theorem for Conditional Expectation). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X_n, X \in L^1_{\mathbb{P}}$ be real-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. If $X_n \rightarrow X$ \mathbb{P} -a.s. and there exists $Y \in L^1_{\mathbb{P}}$ with $|X_n| \leq Y$ (\mathbb{P} -a.s) for all n , then $E[X_n | \mathcal{F}_0] \rightarrow E[X | \mathcal{F}_0]$, \mathbb{P} -a.s.*

Applying this lemma to $X_n = f(Y_n)$ and $X = f(Y)$ we get, in the context of Definition 3, the following property.

Proposition 11.1 (Quenched Convergence with respect to sub sigma-algebras). *If Y_n converges to Y in the quenched sense with respect to \mathcal{F}_0 and $\mathcal{G}_0 \subset \mathcal{F}_0$, then Y_n converges to Y in the quenched sense with respect to \mathcal{G}_0 .*

Note that if Y_n converges to Y in the quenched sense, the convergence in distribution of Y_n to Y is a consequence of Proposition 11.1 by considering $\mathcal{G}_0 = \{\emptyset, \Omega\}$. Though Example 3 shows how the notions of quenched convergence and convergence in distribution differ in general, we will address the problem of non-quenched convergence later, in the specific context of our quenched results. Concretely, we will see that the processes for which the CLT is known to happen within our discussion do not admit a quenched version without a “random centering”, corresponding to the usual normalization of the mean transported to the setting of conditional expectation.

11.2 Regular Conditional Expectations

We begin this section introducing the notion of *regular conditional expectation*, which will allow us to interpret the notion of quenched convergence as a notion of convergence in distribution with respect to a family of measures.

Definition 11.2 (Regular Conditional Expectation). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{F}_0 \subset \mathcal{F}$ be a sub-sigma algebra of \mathcal{F} , and denote by E_0 the conditional expectation with respect to \mathcal{F}_0 . We say that E_0 is **regular** if there exists a family of probability measures $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$ such that for every integrable $X : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}$, the function defined by*

$$\omega \mapsto \int_{\Omega} X(z) d\mathbb{P}_\omega(z) \quad (3.3)$$

*if the integral makes sense², and zero otherwise, defines an \mathcal{F}_0 -measurable version of $E_0 X$. In this case we call $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$ a **decomposition of E_0** .*

Remark 11.2. Note that, in Definition 11.2, X is an actual \mathbb{P} -integrable function, *not* a \mathbb{P} -equivalence class of functions.

Note also that if, in the context of Definition 11.2, X is a bounded function, then the integral in (3.3) is well defined for every $\omega \in \Omega$.

Let now $X \in L^1_{\mathbb{P}}$ be given, and fix a version (also denoted by) X of this function. If $(X_n)_{n \in \mathbb{N}}$ is a family of nonnegative simple functions with $X_n(\omega)$ increasing to $|X(\omega)|$ for all $\omega \in \Omega$ (see for instance p.254 in [11]), then the monotone convergence theorem gives that, for every $\omega \in \Omega$

$$\int_{\Omega} X_n(z) d\mathbb{P}_\omega(z) \rightarrow_n \int_{\Omega} |X(z)| d\mathbb{P}_\omega(z),$$

where the right hand side is regarded as ∞ if $X \notin L^1_{\mathbb{P}_\omega}$.

For every $n \in \mathbb{N}$, denote by \tilde{X}_n the function specified by (3.3) with X_n in place of X . By the definition of $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$, \tilde{X}_n is an \mathcal{F}_0 -measurable version of $E_0 X_n$, and we have just seen that

$$\tilde{X}_n(\omega) \rightarrow_n \int_{\Omega} |X(z)| d\mathbb{P}_\omega(z) \quad (3.4)$$

for every $\omega \in \Omega$.

Now, by Theorem 11.1 and the fact that \tilde{X}_n is a version of $E_0 X_n$,

$$\tilde{X}_n \rightarrow_n E_0 |X|,$$

\mathbb{P} -a.s. This, together with (3.4), implies that

$$E_0[|X|](\omega) = \int_{\Omega} |X(z)| d\mathbb{P}_\omega(z), \quad (3.5)$$

²This will happen over an \mathcal{F}_0 -set of \mathbb{P} -measure one, see Remark 11.2 below.

\mathbb{P} -a.s. In particular,

$$\int_{\Omega} |X(z)| d\mathbb{P}_{\omega}(z) < \infty \quad (3.6)$$

for \mathbb{P} -a.e. ω , provided that the set Ω_X of ω 's where (3.6) holds is \mathcal{F} -measurable. But it turns out that Ω_X is *indeed* \mathcal{F}_0 -measurable: simply note that, following with the notation along this remark

$$\Omega_X := \bigcup_{n \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} [\tilde{X}_k \leq n].$$

Even more is true: denoting by $Y^+ = YI_{[Y \geq 0]}$ and $Y^- = -YI_{[Y < 0]}$ the nonnegative and negative parts of Y , it is easy to see that if $(X_n^+)_{n \in \mathbb{N}}$ and $(X_n^-)_{n \in \mathbb{N}}$ are sequences of simple functions increasing (respectively) to X^+ and X^- then, following the definitions explained above, the function in Definition 11.2 is the same as the function

$$\tilde{X} := \lim_n ((\widetilde{X_n^+} - \widetilde{X_n^-})I_{\Omega_X}),$$

which is clearly \mathcal{F}_0 -measurable and is easily seen to satisfy

$$E[\tilde{X}I_A] = E[XI_A]$$

for every $A \in \mathcal{F}_0$, being therefore a version of E_0X . It follows by linearity that the following proposition holds:

Proposition 11.2. *In the context of Definition 11.2, $\{\mathbb{P}_{\omega}\}_{\omega \in \Omega}$ is a decomposition of E_0 if and only if for every $A \in \mathcal{F}$, $\omega \mapsto \mathbb{P}_{\omega}(A)$ defines an \mathcal{F}_0 -measurable version of $\mathbb{P}[A|\mathcal{F}_0]$.*

Note that, necessarily, the set where the integral makes sense has \mathbb{P} -measure one, because

$$E|X| = EE_0|X| = \int_{\Omega} \int_{\Omega} |X(z)| d\mathbb{P}_{\omega}(z) d\mathbb{P}(\omega)$$

In other words, (3.3) defines an \mathcal{F}_0 -measurable function and $E_0X(\omega) = E^{\omega}X$, \mathbb{P} -a.s., where E^{ω} denotes integration with respect to \mathbb{P}_{ω} . If \mathcal{F} is countably generated and E_0 is regular then the following uniqueness (up to \mathbb{P} -negligible sets) result holds.

Proposition 11.3. *In the context of Definition 11.2, if \mathcal{F} is countably generated and E_0 is regular, and given any two decompositions $\{\mathbb{P}_{1,\omega}\}_{\omega \in \Omega}$ and $\{\mathbb{P}_{2,\omega}\}_{\omega \in \Omega}$ of E_0 , there exists a set $\Omega_0 \subset \Omega$ with $\mathbb{P}\Omega_0 = 1$ such that for every $\omega \in \Omega_0$, $\mathbb{P}_{1,\omega} = \mathbb{P}_{2,\omega}$.*

Proof: Denote by $E_{1,0}$ and $E_{2,0}$ the versions of E_0 given, respectively, by (integration with respect to) $\{\mathbb{P}_{1,\omega}\}_{\omega \in \Omega}$ and $\{\mathbb{P}_{2,\omega}\}_{\omega \in \Omega}$.

Now, given $A \in \mathcal{F}$, consider the function

$$U_A(\omega) := \mathbb{P}_{1,\omega}(A) - \mathbb{P}_{2,\omega}(A) =: E_{1,0}I_A(\omega) - E_{2,0}I_A(\omega). \quad (3.7)$$

Note that U_A is \mathcal{F}_0 -measurable and therefore so is the set $[U_A \geq 0]$. In particular

$$\int_{\Omega} U_A(\omega) I_{[U_A \geq 0]}(\omega) d\mathbb{P}(\omega) =$$

$$\begin{aligned} & \int_{\Omega} E_{1,0} I_A(\omega) I_{[U_A \geq 0]}(\omega) d\mathbb{P}(\omega) - \int_{\Omega} E_{2,0} I_A(\omega) I_{[U_A \geq 0]}(\omega) d\mathbb{P}(\omega) = \\ & \int_{\Omega} E_0[I_A I_{[U_A \geq 0]}](\omega) d\mathbb{P}(\omega) - \int_{\Omega} E_0[I_A I_{[U_A \geq 0]}](\omega) d\mathbb{P}(\omega) = 0, \end{aligned}$$

and by a similar argument using the set $[U_A < 0]$ we conclude that there exists Ω_A with $\mathbb{P}\Omega_A = 1$ such that $\mathbb{P}_{1,\omega}(A) = \mathbb{P}_{2,\omega}(A)$ for every $\omega \in \Omega_A$.

Let $\{A_k\}_{k \in \mathbb{N}}$ be a countable π -system generating \mathcal{F} and let $\Omega_0 := \bigcap_{k \in \mathbb{N}} \Omega_{A_k}$. Clearly, $\mathbb{P}\Omega_0 = 1$.

By the π - λ theorem (applied to the set of $A \in \mathcal{F}$ such that $\mathbb{P}_{1,\omega}(A) = \mathbb{P}_{2,\omega}(A)$ for every $\Omega \in \Omega_0$), $\mathbb{P}_{1,\omega}(A) = \mathbb{P}_{2,\omega}(A)$ for every $A \in \mathcal{F}$ and every $\omega \in \Omega_0$. This is, $\mathbb{P}_{1,\omega} = \mathbb{P}_{2,\omega}$ for every $\omega \in \Omega_0$. \square

Remark 11.3. For future reference, we will point out the following: in the context of Definition 11.2, and given a decomposition $\{\mathbb{P}_{\omega}\}_{\omega \in \Omega}$ of E_0 , a set A satisfies $\mathbb{P}A = 1$ if and only if $\mathbb{P}_{\omega}A = 1$ for \mathbb{P} -a.e. ω . This is a simple consequence of the equality

$$\mathbb{P}A = \int_{\Omega} \mathbb{P}_{\omega}A d\mathbb{P}(\omega).$$

11.3 Regularity and T -Filtrations

Assume that $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ is a given T -filtration (Definition 4.2), that $E_0 := E[\cdot | \mathcal{F}_0]$ is regular, and that $\{\mathbb{P}_{\omega}\}_{\omega \in \Omega}$ is a given decomposition of E_0 . How do we relate the conditional expectations $E[\cdot | \mathcal{F}_k]$ (which depend on \mathbb{P}) with the conditional expectations induced by \mathbb{P}_{ω} ? The following answer is sufficient for our purposes:

Lemma 4. *Let $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ be a T -filtration (Definition 4.2) and for every $k \in \mathbb{Z}$, denote by E_k the conditional expectation with respect to \mathcal{F}_k and \mathbb{P} . Assume that \mathcal{F}_0 is countably generated (Definition 1.5), that E_0 is regular, and that $\{\mathbb{P}_{\omega}\}_{\omega \in \Omega}$ is a decomposition of E_0 (Definition 11.2). Denoting further by E_k^{ω} the conditional expectation with respect to \mathcal{F}_k and \mathbb{P}_{ω} , the following property holds: for every \mathbb{P} -integrable Y , every $k \in \mathbb{Z}$, and every \mathcal{F}_k -measurable version of $E_k Y$, there exists Ω_Y with $\mathbb{P}\Omega_Y = 1$ such that*

$$E_k^{\omega} Y = E_k Y \tag{3.8}$$

\mathbb{P}_{ω} -a.s. for every $\omega \in \Omega_Y$.

Remark 11.4. Note that if Z is any version of $E_k Y$, (3.8) and Remark 11.3 imply that $E_k^{\omega} Y = Z$, \mathbb{P}_{ω} -a.s. for \mathbb{P} -a.e. ω (over a set of probability one depending on Z).

Proof of Lemma 4: Fix a version of $Y \in L^1_{\mathbb{P}}$. We will prove that for any $(\mathcal{F}_k$ -measurable) version of $E_k Y$, there exists a set $\Omega_Y \subset \Omega$ with $\mathbb{P}\Omega_Y = 1$ such that the following holds: for every $\omega \in \Omega_Y$ and every $A \in \mathcal{F}_k$

$$\int_A Y(z) d\mathbb{P}_{\omega}(z) = \int_A E_k Y(z) d\mathbb{P}_{\omega}(z), \tag{3.9}$$

this clearly implies the first conclusion.

Fix a (\mathcal{F}_k) -measurable version of $E_k Y$ and notice that for A fixed, a set $\Omega_{Y,A}$ of probability one such that (3.9) holds for all $\omega \in \Omega_{Y,A}$ exists by the property defining the family $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$ and because

$$E_0[YI_A] = E_0[(E_k Y)I_A],$$

\mathbb{P} -a.s. Without loss of generality $\Omega_{Y,A} \subset \{\omega \in \Omega : |Y| + |E_k Y| \in L^1_{\mathbb{P}_\omega}\}$ (the last set has \mathbb{P} -measure one because $E|Z| = EE_0|Z|$ for every $Z \in L^1_{\mathbb{P}}$).

Now proceed as follows: let $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_k$ be a countable family generating \mathcal{F}_k which is also a π -system and includes Ω (such a family exists because \mathcal{F}_0 is assumed countably generated), let $\Omega_Y := \cap_{n \geq 1} \Omega_{Y,A_n}$, and let $\mathcal{G}_k \subset \mathcal{F}_k$ be the family of sets $A \in \mathcal{F}_k$ such that (3.9) holds for all $\omega \in \Omega_Y$. It is easy to see that \mathcal{G}_k is a λ -system and therefore, since it includes $\{A_n\}_{n \in \mathbb{N}}$, the $\pi - \lambda$ theorem implies that $\mathcal{G}_k = \mathcal{F}_k$. Note that $\mathbb{P}\Omega_{0,Y} = 1$, and that for all $\omega \in \Omega_Y$, (3.9) holds for all $A \in \mathcal{F}_k$.

This gives the proof of the first conclusion. The second conclusion (the one about martingales) follows easily from this, together with the fact that $E|X|^p = EE_0|X|^p$ and therefore $E|X|^p < \infty$ if and only if $E^\omega|X|^p < \infty$ for \mathbb{P} -a.e. ω . \square

Corollary 11.2. *In the context of Lemma 4 and denoting further by E^ω the integration with respect to \mathbb{P}_ω , if $p \geq 1$ and $D_0 \in L^p_{\mathbb{P}}(\mathcal{F}_0)$ is such that $E_{-1}D_0 = 0$, there exists a set $\Omega_0 \subset \Omega$ with $\mathbb{P}\Omega_0 = 1$ such that for every $k \geq 1$ and every $\omega \in \Omega_0$, $E^\omega|T^k D_0|^p < \infty$ and $E_{k-1}^\omega T^k D_0 = 0$, \mathbb{P}_ω -a.s.*

It follows easily that if $(T^k D_0)_{k \in \mathbb{N}}$ is a $(\mathcal{F}_k)_{k \in \mathbb{N}}$ adapted (stationary) sequence of martingale differences in $L^p_{\mathbb{P}}$, then for \mathbb{P} -almost every ω (over a set depending on fixed versions of $(T^k D_0)_{k \in \mathbb{N}}$), $(T^k D_0)_{k \in \mathbb{N}}$ is a $(\mathcal{F}_k)_{k \in \mathbb{N}}$ adapted (not necessarily stationary) sequence of martingale differences in $L^p_{\mathbb{P}_\omega}$.

Proof of Corollary 11.2: Let $D_k := T^k D_0$ ($k \in \mathbb{Z}$), and first note that $E_{k-1}D_k = 0$, \mathbb{P} -a.s. for every $k \in \mathbb{Z}$.

Now let $\Omega_{0,1}$ be a set of probability one such that if $\omega \in \Omega_{0,1}$, $E_{k-1}D_k = E_{k-1}^\omega D_k$ \mathbb{P}_ω -a.s. for all $k \geq 1$ (Lemma 4 and Remark 11.4), let $\Omega_{0,2}$ be a set of probability one with the property that for all $\omega \in \Omega_{0,2}$ and all $k \geq 1$, $E_{k-1}D_k = 0$ \mathbb{P}_ω -a.s. (Remark 11.3), and let $\Omega_{0,3}$ be a set of probability one such that for all $\omega \in \Omega_{0,3}$ and all $k \geq 0$, $E^\omega|D_k|^p < \infty$ (such a set exists because $\infty > E|D_k|^p = E[E_0|D_k|^p]$). The set $\Omega_0 = \cap_{j=1}^3 \Omega_{0,j}$ satisfies the given conclusion. \square

12 Examples of Regularity

In this section we illustrate the notion of regularity by constructing regular conditional expectations in specific settings. The setting in Section 12.2 is of particular interest due to its generality and its importance along the applications.

Let us start by illustrating the trivial cases:

Example 4 (Trivial Examples of Regularity). In the context of Definition 11.2, if \mathcal{F} includes the singletons $\{\omega\}$ ($\omega \in \Omega$), $\mathcal{F}_0 = \mathcal{F}$, and for a given $\omega \in \Omega$, δ_ω denotes the Dirac probability measure at ω ($\delta_\omega\{\omega\} = 1$), then $\{\delta_\omega\}_{\omega \in \Omega}$ is a decomposition of E_0 . If $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is the trivial sigma-algebra, then we get a decomposition $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$ of E_0 by taking $\mathbb{P}_\omega = \mathbb{P}$ for every $\omega \in \Omega$.

12.1 Functions of i.i.d. Sequences

The simplest non-trivial example of a regular conditional expectation is perhaps the following:

Example 5 (Functions of i.i.d. sequences). Refer to the setting explained along Example 1 on page 29 and consider the following observation: if $(\Omega^-, \mathcal{F}^-)$ and $(\Omega^+, \mathcal{F}^+)$ denote respectively the space of complex-valued sequences of the form $(a_k)_{k \leq 0}$ and $(a_k)_{k > 0}$ ($k \in \mathbb{Z}$) with the sigma algebras \mathcal{F}^- and \mathcal{F}^+ generated by the respective finite dimensional cylinders, then $(\mathbb{C}^{\mathbb{Z}}, \mathcal{F}) = (\Omega^- \times \Omega^+, \mathcal{F}^- \otimes \mathcal{F}^+)$ and the projections (defined in an obvious way) $\pi^- : \Omega \rightarrow \Omega^-$ and $\pi^+ : \Omega \rightarrow \Omega^+$ are measurable with respect to the respective sigma-algebras. Note also that $\mathcal{F}_0 = (\pi^-)^{-1}\mathcal{F}^-$.

For every $\omega \in \Omega$, let $\omega^+ := \pi^+(\omega)$ and $\omega^- := \pi^-(\omega)$, consider the function $\delta_\omega : \Omega \rightarrow \Omega$ given by

$$\delta_\omega(z) = (\omega^-, z^+)$$

and define the measure of “partial integration with respect to the future” $\mathbb{P}_\omega := \mathbb{P}\delta_\omega^{-1}$. We claim that if the sigma algebras $\sigma(\xi_k)_{k \leq 0}$ and $\sigma(\xi_k)_{k > 0}$ are independent (with respect to \mathbb{P}) or, equivalently, if $(\xi_k)_{k \in \mathbb{Z}}$ is i.i.d. (consider the hypothesis of stationarity) then $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$ is a decomposition of E_0 .

Let us prove this: first note that, by the hypothesis of independence, $\mathbb{P} = \mathbb{P}^- \times \mathbb{P}^+$, where \mathbb{P}^- (respectively \mathbb{P}^+) is the measure in $(\Omega^-, \mathcal{F}^-)$ (respectively $(\Omega^+, \mathcal{F}^+)$) induced by $(\xi_k)_{k \leq 0}$ (respectively $(\xi_k)_{k > 0}$) by the procedure explained in Example 1.

Now fix $A \in \mathcal{F}$, and let us give an explicit formula for $\mathbb{P}_\omega(A)$:

$$\mathbb{P}_\omega A = \mathbb{P}[\delta_\omega \in A] = \mathbb{P}\{z \in \Omega : (\omega^-, z^+) \in A\} = \mathbb{P}_+\{y \in \Omega^+ : (\omega^-, y) \in A\} \quad (3.10)$$

where we used Fubini’s Theorem (see [11] Theorem 18.3, see also Theorems 18.1 and 18.2 there³) to guarantee the validity of (3.10). By Fubini’s theorem again, the function $u : (\Omega^-, \mathcal{F}^-) \rightarrow [0, \infty)$ given by

$$u(x) = \mathbb{P}^+\{y \in \Omega^+ : (x, y) \in A\} \quad (3.11)$$

is \mathcal{F}^- -measurable. Since $\omega \mapsto \mathbb{P}_\omega(A)$ is the same as $\omega \mapsto u \circ \pi^-(\omega)$, it follows that $\omega \mapsto \mathbb{P}_\omega(A)$ is \mathcal{F}_0 -measurable.

³These are theorems related to real-valued functions, but this poses no serious restriction. The reader may as well replace “ \mathbb{C} ” by “ \mathbb{R} ” in this example and refer to Example 12.2 to cover the complex-valued case.

Now, $\mathcal{F}_0 = \{B \times \Omega^+ : B \in \mathcal{F}^-\}$ (to see this use, for instance, the $\pi - \lambda$ theorem), and a further application of Fubini's theorem shows that for every $B \in \mathcal{F}^-$

$$\begin{aligned} \int_{B \times \Omega^+} \mathbb{P}_\omega(A) d\mathbb{P}(\omega) &= \int_{B \times \Omega^+} \mathbb{P}^+\{y \in \Omega^+ : (\omega^-, y) \in A\} d\mathbb{P}(\omega) = \\ &= \int_B \mathbb{P}^+\{y \in \Omega^+ : (x, y) \in A\} d\mathbb{P}^-(x) = \mathbb{P}(A \cap (B \times \Omega^+)). \end{aligned} \quad (3.12)$$

These facts show that for every $A \in \mathcal{F}$, $\omega \mapsto \mathbb{P}_\omega(A)$ defines a version of $\mathbb{P}[A|\mathcal{F}_0]$, and an application of Proposition 11.2 shows that, indeed, $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$ is a decomposition of E_0 .

12.2 Functions of Stationary Markov Chains

To begin with, let us recall the notion of a transition probability matrix:

Definition 12.1 (Transition Probability Matrix). *Given two measurable spaces (Ω, \mathcal{F}) and $(\mathcal{K}, \mathcal{G})$, a **transition probability matrix** between (Ω, \mathcal{F}) and $(\mathcal{K}, \mathcal{G})$ is a function*

$$P : \Omega \times \mathcal{G} \rightarrow [0, 1] \quad (3.13)$$

*with the property that for every $\omega \in \Omega$, $P(\omega, \cdot)$ is a probability measure in \mathcal{G} and for every $G \in \mathcal{G}$, $P(\cdot, G)$ is \mathcal{F} -measurable. If $(\Omega, \mathcal{F}) = (\mathcal{K}, \mathcal{G})$, we say that P is a transition probability matrix **in** (Ω, \mathcal{F}) .*

We also introduce the following terminology.

Definition 12.2 (Markov Chains). *Assume that for every $k \in \mathbb{Z}$, a measurable space (S_k, \mathcal{S}_k) is given, and let $(\xi_k)_{k \in \mathbb{Z}}$ be a sequence of random elements $\xi_k : \Omega \rightarrow S_k$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (Definition 2.3). We say that $(\xi_k)_{k \in \mathbb{Z}}$ is a **Markov chain** if for every $k \in \mathbb{Z}$ there exists a transition probability matrix (Definition 12.1) $P_k : S_k \times \mathcal{S}_{k+1} \rightarrow [0, 1]$ such that for every $A \in \mathcal{S}_{k+1}$,*

$$\omega \mapsto P_k(\xi_k(\omega), A) \quad (3.14)$$

*defines a version of $\mathbb{P}[\xi_{k+1} \in A | \sigma(\xi_j)_{j \leq k}] := E[I_A \circ \xi_{k+1} | \sigma(\xi_j)_{j \leq k}]$ (the conditional expectation is taken with respect to \mathbb{P}). The Markov chain has a **fixed state space** if $(S_k, \mathcal{S}_k) = (S_0, \mathcal{S}_0)$ for every $k \in \mathbb{Z}$. If the state space is fixed, the Markov chain is **stationary** if $(\xi_k)_{k \in \mathbb{Z}}$ is stationary (the law of $(\xi_n, \dots, \xi_{n+k})$ is the same for every $n \in \mathbb{Z}$ if $k \in \mathbb{Z}$ is fixed), and it is **homogeneous** if (we can choose) $P_k = P_0$ for all $k \in \mathbb{Z}$.*

Remark 12.1. Every stationary Markov chain is homogeneous: in the context of Definition 12.2, given $k \in \mathbb{Z}$ and $A, B \in \mathcal{S} := \mathcal{S}_0$,

$$\begin{aligned} \mathbb{P}([\xi_{k+1} \in A] \cap [\xi_k \in B]) &= \mathbb{P}([\xi_1 \in A] \cap [\xi_0 \in B]) = \int_{\Omega} P_0(\xi_0(\omega), A) I_B(\xi_0(\omega)) d\mathbb{P}(\omega) = \\ &= \int_S P_0(x, A) I_B(x) d\mathbb{P}_{\xi_0}^{-1}(x) = \int_S P_0(x, A) I_B(x) d\mathbb{P}_{\xi_k}^{-1}(x) = \\ &= \int_{\Omega} P_0(\xi_k(\omega), A) I_B(\xi_k(\omega)) d\mathbb{P}(\omega), \end{aligned}$$

so that, necessarily, $P_k(\xi_k(\omega), A) = P_0(\xi_k(\omega), A)$, \mathbb{P} -a.s. And we can replace $P_k = P_0$.

Note also that condition (3.14) implies, in particular, that for every $A \in \mathcal{S}$ and every $k \in \mathbb{Z}$

$$\mathbb{P}[\xi_{k+1} \in A | \sigma(\xi_j)_{j \leq k}] = \mathbb{P}[\xi_{k+1} \in A | \sigma(\xi_k)]. \quad (3.15)$$

Definition 12.3 (Generalized Markov Chain). *In the context of Definition 12.2, if we can verify (3.15) (regardless of whether the family of transitions matrices $(P_k)_{k \in \mathbb{Z}}$ satisfying (3.14) exists), we call $(\xi_k)_{k \in \mathbb{Z}}$ is a **generalized Markov chain**. The generalized Markov chain has a fixed state space if for every $k \in \mathbb{Z}$, $(S_k, \mathcal{S}_k) = (S_0, \mathcal{S}_0)$.*

Every generalized Markov chain whose state space is a complete and separable metric space (with its Borel sigma algebra) is a Markov chain:

Proposition 12.1 (Existence of Markov Kernels). *If (S, d) is a complete and separable metric space with Borel sigma algebra \mathcal{S} and ξ_1, ξ_2 are random elements on S defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, there exists a transition probability matrix P in (S, \mathcal{S}) (Definition 12.1) such that the map $\Omega \rightarrow [0, 1]$ given by*

$$\omega \mapsto P(\xi_1(\omega), A)$$

defines a version of $\mathbb{P}[\xi_2 \in A | \sigma(\xi_1)]$.

Proof: This follows from Exercise 1 in [7], Section 44. □

Our first example in this section is the following:

Example 6 (Functions of Stationary Markov Chains). To motivate the construction that follows, start by noticing that Example 5 can be extended to the context in which $\sigma(\xi_k)_{k \leq 0}$ and $\sigma(\xi_k)_{k \geq 0}$ are not necessarily independent, provided that if we replace \mathbb{P}^+ by a measure \mathbb{P}^x in (3.11) the function $u(x)$ is (still) \mathcal{F}^- -measurable, and that we can (still) verify the equalities in (3.12) (dropping the third term) with \mathbb{P}^+ replaced by \mathbb{P}^x .

This is the case for instance in the context of *functions of stationary, homogeneous Markov Chains on a complete and separable metric space S* . In what follows (S, d) denotes a complete and separable metric space with metric d and Borel sigma algebra \mathcal{S} and, for every $k \in \mathbb{N}^*$, \mathcal{S}^k denotes the product sigma algebra in S^k . The rest of the notation copies that in Example 1 and Section 12.1: we will work (again) over the space (Ω, \mathcal{F}) of S -valued sequences $(a_k)_{k \in \mathbb{Z}}$ with the sigma algebra generated by the finite-dimensional cylinders, and we will use the decomposition $(\Omega, \mathcal{F}) = (\Omega^- \times \Omega^+, \mathcal{F}^- \otimes \mathcal{F}^+)$ as in Section 12.1 (with \mathbb{C} replaced by S). Again, $\mathcal{F}_0 = \sigma(x_k)_{k \leq 0}$, where $x_j : \Omega \rightarrow S$ is the projection in the j -th coordinate.

Let us start by explaining the construction of the processes under consideration.

Construction of stationary, homogeneous Markov Chains

Let $P : S \times \mathcal{S} \rightarrow [0, 1]$ be a transition probability matrix in (S, \mathcal{S}) (Definition 12.1). We will also assume that $P(\cdot, \cdot)$ admits an invariant probability measure \mathbb{P} (see item 6. below). Our goal is to construct a probability measure $\mathbb{P}_{\mathbb{Z}}$ on $(S^{\mathbb{Z}}, \mathcal{S}^{\mathbb{Z}})$ such that the

coordinate functions $x_k : S^{\mathbb{Z}} \rightarrow S$ define a stationary Markov Chain (with $\mathbb{P}_{\mathbb{Z}}x_0^{-1} = \mathbb{P}$) on $(S^{\mathbb{Z}}, \mathcal{S}^{\mathbb{Z}}, \mathbb{P}_{\mathbb{Z}})$ with state space S and transition probability P : for every $k \in \mathbb{Z}$,

$$\mathbb{P}_{\mathbb{Z}}(x_{k+1} \in A | x_k) = P(x_k, A).$$

The construction can be summarized as follows:

1. Given $A_0, A_1, \dots, A_k \in \mathcal{S}$, $x \in S$ and $n \in \mathbb{Z}$, and denoting the integral of a measurable function $f : S \rightarrow \mathbb{C}$ with respect to the measure $P(x, \cdot)$ by

$$\int_S f(y) P(x, dy)$$

define $P_0^k(x, A_0 \times \dots \times A_n)$ by

$$\begin{aligned} P_0^k(x, A_0 \times \dots \times A_n) &:= I_{A_0}(x) \int_{A_1} \dots \int_{A_{k-1}} P(y_{k-1}, A_k) P(y_{k-2}, dy_{k-1}) \dots P(x, dy_1) \\ &= \int_S \dots \int_S I_{A_0 \times A_1 \times \dots \times A_k}(x, y_1, \dots, y_k) P(y_{k-1}, dy_k) \dots P(x, dy_1). \end{aligned} \quad (3.16)$$

The $\pi - \lambda$ theorem and Kolmogorov's existence theorem⁴ guarantee that $P_0^k(x, \cdot)$, thus defined over rectangular sets, extends in a unique way to a measure $P_0^\infty(x, \cdot)$ on $\mathcal{S}^{\mathbb{N}}$ (more explicitly: Theorem 3.1 in [11] allows us to see that (3.16) defines a unique probability measure in \mathcal{S}^{k+1} , and the $\pi - \lambda$ theorem guarantees that if $A \in \mathcal{S}^k$, $P_0^k(x, A \times S) = P_0^{k-1}(x, A)$. An application of Proposition III-3-3 in [40] implies that $P_0^\infty(x, \cdot)$ exists and is unique). Even more (see Proposition V-2-1 in [40]), for every $A \in \mathcal{S}^{\mathbb{N}}$, the function $S \rightarrow [0, 1]$ given by

$$x \mapsto P_0^\infty(x, A)$$

is \mathcal{S} -measurable. Thus $P_0^\infty : S \times \mathcal{F}^+ \rightarrow [0, 1]$ is a transition probability between (S, \mathcal{S}) and $(\Omega^+, \mathcal{F}^+)$.

2. We can extend P_0^∞ to a transition probability between $(\Omega^-, \mathcal{F}^-)$ and $(\Omega^+, \mathcal{F}^+)$ in the following way: given $\omega^- \in \Omega^-$ and $A \in \mathcal{F}^+$

$$\mathbb{P}_0^\infty(\omega^-, A) := P_0^\infty(x_0(\omega^-), A). \quad (3.17)$$

\mathbb{P}_0^∞ , thus extended, is clearly \mathcal{F}^- measurable for every fixed A , showing that it is (indeed) a transition probability matrix $\Omega^- \times \mathcal{F}^+ \rightarrow [0, 1]$.

3. Notice that for every $k \in \mathbb{N}^*$, P_0^∞ restricts to a transition probability $P_0^k : S \times \mathcal{S}^k \rightarrow [0, 1]$ in the obvious way: if $\pi_k : S^{\mathbb{N}} \rightarrow S^k$ is the natural projection, $P_0^k(x, A) := P_0^\infty(x, \pi_k^{-1}(A))$.

More explicitly, note that $P_0^1(x, A) = P(x, A)$ and for general k , $P_0^k(x, A)$ is given by the last line of (3.16) with $I_{A_0 \times \dots \times A_k}$ replaced by I_A (apply the $\pi - \lambda$ theorem to the λ -system of sets in \mathcal{S}^{k+1} for which this holds).

⁴It is important to point out that Kolmogorov's existence theorem is not guaranteed without special assumptions on the structure of the underlying measurable space (see [1] for counterexamples). The validity of Kolmogorov's existence theorem for the case of complete and separable metric spaces is, on the other side, a well established fact.

4. Given a probability measure μ on \mathcal{S} , $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, define the probability measure μ_n^{n+k} in \mathcal{S}^k in the following way: for every $A \in \mathcal{S}^k$

$$\mu_n^{n+k} A := \int_S P_0^k(x, A) d\mu(x).$$

5. If we assume that \mathbb{P} , a probability measure on \mathcal{S} , is a *stationary probability measure* for P . This is, that $\mathbb{P}_0^1 = \mathbb{P}$, then it is easy to see that for every $n \in \mathbb{Z}$, $k \in \mathbb{N}$ and $A \in \mathcal{S}$,

$$\mathbb{P}_n^{n+k}(S \times \cdots \times S \times A) = \mathbb{P}(A). \quad (3.18)$$

It follows from this that for every simple function $f(y) = \sum_{j=1}^r a_j I_{A_j}$ ($A_j \in \mathcal{S}$):

$$\int_S \cdots \int_S f(y) P(y_{n-1}, y) \cdots P(x, y_{n-l+1}) d\mathbb{P}(x) = \int_S f(x) d\mathbb{P}(x)$$

and by an approximation argument analogous to the one leading to Proposition 11.2 the same holds for every \mathbb{P} -integrable function f .

6. In particular the following holds: for every $A_n, \dots, A_{n+k} \in \mathcal{S}$, if we denote

$$f(y) := \int_S \cdots \int_S I_{A_n \times \cdots \times A_{n+k}}(y, y_{n+1}, \dots, y_{n+k}) P(y_{n+k-1}, dy_{n+k}) \cdots P(y, dy_{n-1})$$

then

$$\begin{aligned} \mathbb{P}_{n-l}^{n+k}(S^l \times A_n \times \cdots \times A_{n+k}) &= \int_S \cdots \int_S f(y_n) P(y_{n-1}, y_n) \cdots P(x, y_{n-l+1}) d\mathbb{P}(x) = \\ &= \int_S f(x) d\mathbb{P}(x) = \int_S P_0^k(x, A_n \times \cdots \times A_{n+k}) d\mathbb{P}(x) =: \mathbb{P}_0^k(A_n \times \cdots \times A_{n+k}), \end{aligned}$$

and it follows by a further application of the $\pi - \lambda$ theorem and Kolmogorov's existence theorem that there exists a unique probability measure $\mathbb{P}_{\mathbb{Z}}$ on (Ω, \mathcal{F}) such that for every $k \in \mathbb{N}$ and every set of the form

$$H_{k,A} := [(x_{-k}, \dots, x_k) \in A] \quad (3.19)$$

where $A \in \mathcal{S}^{2k+1}$,

$$\mathbb{P}_{\mathbb{Z}}(H_{k,A}) = \mathbb{P}_0^{2k+1}(A)$$

(note that the sets of the form (3.19) indeed generate \mathcal{F}).

The coordinate functions $(x_k)_{k \in \mathbb{Z}}$ give, in this setting, a stationary Markov chain defined on $(\Omega, \mathcal{F}, \mathbb{P}_{\mathbb{Z}})$ with transition probability P and law \mathbb{P} (see [40], V-2 for more details on this). It is not hard to see in particular that for every $n \in \mathbb{Z}$, $k \in \mathbb{N}$ and $f \in L_{\mathbb{P}}^1$,

$$E[f(x_{n+k}) | \sigma(x_n)](\omega) = \int_S f(y) P_0^k(x_n(\omega), dy),$$

where $P_0^k(x, dy)$ denotes (in this case) the marginal distribution

$$P_0^k(x, A) = P_0^k(x, S \times \cdots \times S \times A),$$

for $A \in \mathcal{S}$. More generally, given any function $f : S^k \rightarrow \mathbb{C}$ such that $f \circ (x_1, \dots, x_k)$ is $\mathbb{P}_{\mathbb{Z}}$ -integrable:

$$E[f \circ (x_{n+1}, \dots, x_{n+k}) | \sigma(x_n)](\omega) = \int_S \cdots \int_S f(z_1, \dots, z_k) P(z_{k-1}, dz_k) \cdots P(x_n(\omega), dz_1). \quad (3.20)$$

For future reference, we will introduce the notation

$$(P_0^k f)(z_0) := \int_S \cdots \int_S f(z_1, \dots, z_k) P(z_{k-1}, dz_k), \dots, P(z_0, dz_1). \quad (3.21)$$

where $f : S^k \rightarrow \mathbb{C}$ is an appropriate function (in particular $E[f \circ (x_{n+1}, \dots, x_{n+k}) | \sigma(x_n)] = P_0^k f \circ x_n$).

Regularity

We will show now that, again, $E_0 = E[\cdot | \mathcal{F}_0]$ is regular.

1. To do so we proceed as follows: given $\omega \in \Omega$, let \mathbb{P}_ω be the probability measure on $\mathcal{F} = \mathcal{F}^- \otimes \mathcal{F}^+$ given in the following way: for $A \in \mathcal{F}$,

$$\mathbb{P}_\omega(A) = \mathbb{P}_0^\infty(\omega^-, \{y \in \Omega^+ : (\omega^-, y) \in A\}) \quad (3.22)$$

where \mathbb{P}_0^∞ is given by (3.17). We proceed now to verify that for every $A \in \mathcal{F}$, $\omega \mapsto \mathbb{P}_\omega(A)$ is a version of $\mathbb{P}_{\mathbb{Z}}[A | \mathcal{F}_0]$ which (again) is sufficient to prove the regularity of $E[\cdot | \mathcal{F}_0]$ in virtue of Proposition 11.2.

2. *\mathcal{F}_0 -measurability.* To see that $\omega \mapsto \mathbb{P}_\omega(A)$ is \mathcal{F}_0 -measurable note that, by the $\pi - \lambda$ theorem applied to the λ -system of sets $A \in \mathcal{F}$ such that $\omega \mapsto \mathbb{P}_\omega(A)$ is \mathcal{F}_0 -measurable, it suffices to see that this is the case under the assumption that $A = A^- \times A^+ \in \mathcal{F}^- \times \mathcal{F}^+$. But it is easy to see that, in this case

$$\mathbb{P}_\omega(A) = I_{A^-}(\omega^-) \mathbb{P}_0^\infty(\omega^-, A^+)$$

which defines an \mathcal{F}_0 -measurable function of ω because the function $f_A : (\Omega^-, \mathcal{F}^-) \rightarrow [0, 1]$ given by

$$f_A(u) = I_{A^-}(u) \mathbb{P}_0^\infty(u, A^+)$$

is \mathcal{F}^- -measurable and

$$\mathbb{P}_\omega(A) = f_A \circ \pi^-(\omega).$$

3. *Integral equation.* To check that for every $A \in \mathcal{F}_0$ and $B \in \mathcal{F}$

$$\int_\Omega \mathbb{P}_\omega(B) I_A(\omega) d\mathbb{P}_{\mathbb{Z}}(\omega) = \mathbb{P}_{\mathbb{Z}}(A \cap B) \quad (3.23)$$

we start by noticing the following: if we can check (3.23) for

$$A' = [(x_{-k}, \dots, x_0) \in A'_{-k} \times \cdots \times A'_0] \quad (3.24)$$

fixed and every set B of the form

$$B = [(x_{-l}, \dots, x_l) \in A_{-l} \times \dots \times A_0 \times B_1 \times \dots \times B_l] \quad (3.25)$$

then, by the $\pi - \lambda$ theorem, (3.23) holds for every $B \in \mathcal{F}$ whenever A' is a finite dimensional cylinder of the form (3.24). Then, since for fixed $B \in \mathcal{F}$, (3.23) holds for every finite dimensional cylinder A' of the form (3.24) and these generate \mathcal{F}_0 , a new application of the $\pi - \lambda$ theorem gives (3.23) for every $A \in \mathcal{F}_0$.

4. Thus it suffices to check (3.23) for A' , B as in (3.24) and (3.25). Note that without loss of generality we can assume that $k = l$, and that in this case, taking $C_j := A_j \cap A'_j$ ($j = -k, \dots, 0$),

$$\mathbb{P}_\omega(B)I_{A'}(\omega) = I_{C_{-k} \times \dots \times C_0}(x_{-k}(\omega), \dots, x_0(\omega))P_0^k(x_0(\omega), B_1 \times \dots \times B_k) = \mathbb{P}_\omega(C),$$

where

$$C = [(x_{-k}, \dots, x_k) \in C_{-k} \times \dots \times C_0 \times B_1 \times \dots \times B_k].$$

In conclusion, it suffices to see that if B is any cylinder of the form (3.25):

$$\int_{\Omega} \mathbb{P}_\omega(B) d\mathbb{P}_{\mathbb{Z}}(\omega) = \mathbb{P}_{\mathbb{Z}}(B).$$

Let us do this for the case $k = 1$ (the general case is analogous):

$$\begin{aligned} \int_{\Omega} \mathbb{P}_\omega(B) d\mathbb{P}_{\mathbb{Z}}(\omega) &= \int_{\Omega} I_{A_{-1} \times A_0}(x_{-1}(\omega), x_0(\omega))P_0^1(x_0(\omega), B_1) d\mathbb{P}_{\mathbb{Z}}(\omega) = \\ &= \int_{\Omega} \left(\int_S I_{A_{-1} \times A_0 \times B_1}(x_{-1}(\omega), x_0(\omega), z_1) P(x_0(\omega), dz_1) \right) d\mathbb{P}_{\mathbb{Z}}(\omega) = \\ &= \int_S \int_S \int_S I_{A_{-1} \times A_0 \times B_1}(z_{-1}, z_0, z_1) P(z_0, dz_1) P(z_{-1}, dz_0) d\mathbb{P}(z_{-1}) = \mathbb{P}(B) \end{aligned}$$

as desired.

The result of this construction can be summarized in the following way:

Proposition 12.2 (Functions of Stationary Markov Chains and Regular Conditional Expectations). *If $(\Omega, \mathcal{F}, \mathbb{P}_{\mathbb{Z}})$ is the probability space constructed above, $T : \Omega \rightarrow \Omega$ is the left shift (specified again by $x_k \circ T = x_{k+1}$) and for some $p \geq 1$, $f : \Omega^- \rightarrow \mathbb{C}$ belongs to $L^p_{\mathbb{P}_{\mathbb{Z}}}$ (where f is extended to Ω in the obvious way: $\tilde{f}(\omega) = f(\omega^-)$), then*

- (a). *If $\mathcal{F}_0 := \sigma(x_k)_{k \leq 0}$, then $(\mathcal{F}_k)_{k \in \mathbb{Z}} := (T^k \mathcal{F}_0)_{k \in \mathbb{Z}} = (\sigma(x_j)_{j \leq k})_{k \in \mathbb{Z}}$ is a T -filtration.*
- (b). *If for every $k \in \mathbb{Z}$, $X_k := T^k f := f \circ T^k$, then the stationary sequence $(X_k)_{k \in \mathbb{Z}}$ is $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ -adapted.*
- (c). *The conditional expectation $E_0 = E[\cdot | \mathcal{F}_0]$ is regular, and for every $X \in L^1_{\mathbb{P}_{\mathbb{Z}}}$, a version of $E_0 X$ is given by*

$$E_0[X](\omega) = \int_{\Omega} X(z) d\mathbb{P}_{\omega}(z)$$

where, for every $\omega \in \Omega$, \mathbb{P}_{ω} is given by (3.22).

Our last example shows how to represent a stationary sequence of random functions (on a complete and separable metric space) as a function of a Markov chain, a construction that allows us to see that a stationary process admits regular conditional expectations with respect to “the past”.

Example 7 (Stationary Sequences as Functions of Markov Chains). Under the setting introduced in Example 6, consider now the $\mathcal{F}/\mathcal{F}^-$ measurable function $\xi_0 = \pi^-$ and the $\mathcal{F}^-/\mathcal{S}$ measurable function x_0^- : the restriction of x_0 to Ω^- . Note that, if for every $k \in \mathbb{Z}$, $\xi_k := T^k \xi_0$ then, since $\sigma(\xi_0) = \sigma(x_j)_{j \leq 0} = \mathcal{F}_0$, we have that for every $k \in \mathbb{Z}$, $\sigma(\xi_k) =: \mathcal{F}_k$ and therefore, since $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ is increasing, $\sigma((\xi_j)_{j \leq k}) = \sigma(\xi_k)$.

In particular, if \mathbb{P} is any probability measure in (Ω, \mathcal{F}) , then for any $k \in \mathbb{Z}$ and any $\mathcal{F}^-/\mathcal{C}$ measurable function $f : \Omega^- \rightarrow \mathbb{C}$

$$\mathbb{P}[f(\xi_{k+1})|\sigma(\xi_j)_{j \leq k}] = \mathbb{P}[f(\xi_{k+1})|\sigma(\xi_k)], \quad (3.26)$$

provided that $f(\xi_{k+1}) \in L^1_{\mathbb{P}}(\mathcal{F})$. By taking $f = I_A$ for any given $A \in \mathcal{F}^-$ we see, by an application of Proposition 12.1, that (3.26) implies that $(\xi_k)_{k \in \mathbb{Z}}$ is a Markov chain⁵ under (any) \mathbb{P} . If $(\xi_k)_{k \in \mathbb{Z}}$ is stationary (under \mathbb{P}) it is also homogeneous (Remark 12.1).

Let now $(X'_k)_{k \in \mathbb{Z}}$ be a sequence of random elements in S defined on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. Assume that $(X'_k)_{k \in \mathbb{Z}}$ is stationary, so that the probability measure on $\mathcal{S}^{\mathbb{Z}}$ specified by

$$\mathbb{P}_{\mathbb{Z}}((x_n, \dots, x_{n+k}) \in A) := \mathbb{P}'((X'_n, \dots, X'_{n+k}) \in A)$$

for every $A \in \mathcal{S}^{k+1}$ makes $(x_k)_{k \in \mathbb{Z}}$ a copy (in distribution) of $(X_k)_{k \in \mathbb{Z}}$. Under $\mathbb{P}_{\mathbb{Z}}$, the Markov chain $(\xi_k)_{k \in \mathbb{Z}}$ is stationary.

If we apply the previous observations to $\mathbb{P}_{\mathbb{Z}}$, and consider $f := x_0^-$ we get that, for every k , $f(T^k \xi_0) = f(\xi_k) = x_k$, and therefore $x_k = f(\xi_k)$ is a *function* of the (stationary) Markov chain $(\xi_k)_{k \in \mathbb{Z}}$. Since the finite dimensional distributions of $(x_k)_{k \in \mathbb{Z}}$ (under $\mathbb{P}_{\mathbb{Z}}$) are the same as those of X'_k (under \mathbb{P}'), we see that *every stationary process in a complete and separable metric space is equivalent (in distribution) to a function of a Markov chain*. Under this equivalence, the “past” sigma algebra is regular: there exists a family of probability measures $\{\mathbb{P}_{\omega}\}_{\omega \in \Omega}$ such that, under $\mathbb{P}_{\mathbb{Z}}$

$$E[x_k|\sigma(x_j)_{j \leq 0}](\omega) = E[f(\xi_k)|\sigma(\xi_0)](\omega) = \int_{\Omega} f(z) d\mathbb{P}_{\omega}(z).$$

As a matter of fact, the analysis in Example 6 shows that

$$E[f(\xi_k)|\sigma(\xi_0)](\omega) = P_0^k f(\xi_0(\omega)),$$

where $P_0^k f$ is given by (3.21) (considering f as constant in (z_1, \dots, z_{k-1})) via the transition probability matrix guaranteed for $(\xi_k)_{k \in \mathbb{Z}}$ by Proposition 12.1 and Remark 12.1.

⁵The state space $(\Omega^-, \mathcal{F}^-)$ is generated by a complete and separable metric space by the standard fact that the countable product of such spaces can be metrized in such a way that it has those two properties.

Remark 12.2 (Nonstationary Case). An analysis of the arguments in Example 6 shows that the assumption of stationarity is superfluous in the following sense: the role of \mathbb{P} (the invariant measure of P) in the construction carried along that example is to guarantee that the coordinate functions *indeed* define a stationary process *and* that the compatibility conditions of Kolmogorov's existence theorem hold. We can drop the requirement of stationarity and still carry on the given construction if we start from "marginal" probability measures $\{\mathbb{P}_n\}_{n \in \mathbb{Z}}$ on \mathcal{S} (representing the distribution of ξ_n), a family of transition measures $\{P_n\}_{n \in \mathbb{Z}}$ on $(\mathcal{S}, \mathcal{S})$ (representing the transitions $\mathbb{P}(\xi_{n+1} \in A | \sigma(\xi_n))$) and if, following the arguments in Example 6, (3.18) holds for every $(n, k) \in \mathbb{Z} \times \mathbb{N}$ if we replace \mathbb{P} by \mathbb{P}_{n+k} .

This is the case if $(X'_k)_{k \in \mathbb{Z}}$ is *any* sequence of random elements in a complete and separable metric space \mathcal{S} and \mathbb{P}_n is the law of $\xi_n = (\cdots, X'_{n-1}, X'_n)$ in $(\Omega^-, \mathcal{F}^-)$. By following the construction along Example 7, this gives a representation of any sequence of random elements on $(\mathcal{S}, \mathcal{S})$ as a sequence of functions of a (not necessarily stationary) Markov chain.

13 Regularity and Quenched Convergence

The natural question at this point is the following: suppose, in the context of Definition 11.2, that E_0 is regular, and assume that X_n converges in the quenched sense to X as $n \rightarrow \infty$. Can we say anything about the convergence of $(X_n)_{n \in \mathbb{N}}$ with respect to the measures in the decomposition of E_0 ? The following proposition provides an answer sufficiently good for our purposes.

Proposition 13.1 (Regularity and Quenched Convergence). *In the context of Definition 11.1, assume that (\mathcal{S}, d) is separable. If E_0 is regular and Y_n converges to Y in the quenched sense, there exists a set $\Omega_0 \subset \Omega$ with $\mathbb{P}\Omega_0 = 1$ such that for all $f : \mathcal{S} \rightarrow \mathbb{R}$ continuous and bounded and all $\omega \in \Omega_0$*

$$\int_{\Omega} f \circ Y_n(z) d\mathbb{P}_{\omega}(z) \rightarrow_n \int_{\Omega'} f \circ Y(z) d\mathbb{P}'(z). \quad (3.27)$$

In particular, Y_n converges to Y in the quenched sense if and only if for \mathbb{P} -a.e ω , $Y_n \Rightarrow Y$ with respect to \mathbb{P}_{ω} .

Notice that if $\{\mathbb{P}_{\omega}\}_{\omega \in \Omega}$ is a decomposition of E_0 then, by the definition of quenched convergence, and denoting again by E^{ω} the integration with respect to \mathbb{P}_{ω} ,

$$E^{\omega} f(Y_n) \rightarrow E f(Y)$$

as $n \rightarrow \infty$ for every $\omega \in \Omega_f$, where $\mathbb{P}\Omega_f = 1$. Proposition 13.1 states that if (\mathcal{S}, d) is separable, Ω_f can be chosen *independent* of f , namely $\Omega_f := \Omega_0$ for all $f \in \mathbf{C}^b(\mathcal{S})$. The set Ω_0 depends, nonetheless, on $(Y_n)_n$.

Proof of Proposition 13.1: Consider functions $U_{k,\epsilon}$ as in the statement 2. of Theorem 6.1. As remarked in the paragraph above there exists, for all $k \in \mathbb{N}$ and $\epsilon > 0$ ($\epsilon \in \mathbb{Q}$),

a set $\Omega_{k,\epsilon} \subset \Omega$ with $\mathbb{P}\Omega_{k,\epsilon} = 1$ such that for all $\omega \in \Omega_{k,\epsilon}$, $E^\omega U_{k,\epsilon}(Y_n) \rightarrow EU_{k,\epsilon}(Y)$ as $n \rightarrow \infty$. Now take $\Omega_0 := \bigcap_{k,\epsilon} \Omega_{k,\epsilon}$ and use Theorem 6.1. \square

14 Product Spaces and Regularity

We finish our discussion about regular conditional expectations with the following result, showing that the notion of regular conditional expectation behaves well under the product of probability spaces.

Proposition 14.1 (Product Spaces and Regularity). *Let $(\Theta, \mathcal{B}, \lambda)$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be probability spaces, and let $\mathcal{B}_0 \subset \mathcal{B}$, $\mathcal{F}_0 \subset \mathcal{F}$ be sub-sigma algebras such that $E[\cdot | \mathcal{B}_0]$ and $E[\cdot | \mathcal{F}_0]$ are regular (Definition 11.2). If $\{\lambda_\theta\}_{\theta \in \Theta}$ and $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$ are, respectively, decompositions of $E[\cdot | \mathcal{B}_0]$ and $E[\cdot | \mathcal{F}_0]$, then the conditional expectation $E[\cdot | \mathcal{B}_0 \otimes \mathcal{F}_0]$ with respect to $\mathcal{B}_0 \otimes \mathcal{F}_0$ and $\lambda \times \mathbb{P}$ admits the decomposition $\{\lambda_\theta \times \mathbb{P}_\omega\}_{(\theta,\omega) \in \Theta \times \Omega}$.*

Proof: We will proceed in two steps.

Step 1. Assume that $\mathcal{B}_0 = \mathcal{B}$. In this case we will prove that for any $E \in \mathcal{B} \otimes \mathcal{F}$ the function

$$\tilde{I}_E(\theta, \omega) = \int_{\Omega} I_E(\theta, z) d\mathbb{P}_\omega(z) \quad (3.28)$$

(which is well defined for every θ by Theorem 18.1 in [11]) defines a version of $\mathbb{P}[E | \mathcal{B} \otimes \mathcal{F}_0]$. Note that by Proposition 11.2 this proves also the desired conclusion for any $f \in L^1_{\lambda \times \mathbb{P}}$, and that in the special case in which for every $\theta \in \Theta$, $\{\theta\} \in \mathcal{B}$ (and therefore $\lambda_\theta = \delta_\theta$, the Dirac measure at θ , defines a decomposition for $E[\cdot | \mathcal{B}] = Id$, the identity map on L^1_λ) there is consistency with the given conclusion.

To prove that (3.28) defines a $\mathcal{B} \otimes \mathcal{F}_0$ -measurable function, note first that if $E = A \times B$ is a rectangular set, then (3.28) is equal to the function

$$(\theta, \omega) \mapsto I_A(\theta) \mathbb{P}_\omega(B),$$

which is clearly $\mathcal{B} \otimes \mathcal{F}_0$ measurable.

Now consider the family \mathcal{G} of sets $E \in \mathcal{B} \otimes \mathcal{F}$ such that (3.28) is $\mathcal{B} \otimes \mathcal{F}_0$ -measurable. Since for any family $\{E_n\}_n \subset \mathcal{G}$ of mutually disjoint sets the choice $E = \cup_n E_n$ gives that

$$\tilde{I}_E(\theta, \omega) = \sum_n \tilde{I}_{E_n}(\theta, \omega)$$

(apply the monotone convergence theorem) and \mathcal{G} includes the set $\Theta \times \Omega$, \mathcal{G} is a λ -system. Since \mathcal{G} includes the finite unions of disjoint rectangles it follows, by the $\pi - \lambda$ theorem, that $\mathcal{G} = \mathcal{B} \otimes \mathcal{F}$. This proves the $\mathcal{B} \otimes \mathcal{F}_0$ -measurability of (3.28) for every $E \in \mathcal{B} \otimes \mathcal{F}$.

Now, by Fubini's theorem and the definition of $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$, given any rectangular set $E' = A' \times B' \in \mathcal{B} \otimes \mathcal{F}_0$

$$\int_{\Theta \times \Omega} \tilde{I}_E(\theta, \omega) I_{E'}(\theta, \omega) d(\lambda \times \mathbb{P})(\theta, \omega) = \int_{A'} \int_{B'} \tilde{I}_E(\theta, \omega) d\mathbb{P}(\omega) d\lambda(\theta) =$$

$$\int_{A'} \int_{B'} I_E(\theta, \omega) d\mathbb{P}(\omega) d\lambda(\theta) = (\lambda \times \mathbb{P})(E \cap E'),$$

and a further application of the $\pi - \lambda$ theorem shows that the equality between the extremes holds for any $E' \in \mathcal{B} \otimes \mathcal{F}_0$, which proves that (3.28) is indeed a version of $\mathbb{P}[E|\mathcal{B} \otimes \mathcal{F}_0]$.

Step 2. General Case. For the general case note first the following: by the case treated in the previous step and Proposition 11.2, given any $f \in L^1_{\lambda \times \mathbb{P}}$ the function

$$\tilde{f}(\theta, \omega) = \int_{\Omega} f(\theta, z) d\mathbb{P}_{\omega}(z)$$

is a version of $E[f|\mathcal{B} \otimes \mathcal{F}_0]$. Also

$$E[f|\mathcal{B}_0 \otimes \mathcal{F}_0] = E[E[f|\mathcal{B} \otimes \mathcal{F}_0]|\mathcal{B}_0 \otimes \mathcal{F}_0] = E[\tilde{f}|\mathcal{B}_0 \otimes \mathcal{F}_0]$$

$(\lambda \times \mathbb{P})$ -a.s. It follows by a second application of the Step 1 and Proposition 11.2 with $\mathcal{B} \otimes \mathcal{F}_0$ in the role of $\mathcal{B} \otimes \mathcal{F}$ and with \mathcal{B}_0 in the role of \mathcal{F}_0 , that

$$(\theta, \omega) \mapsto \int_{\Theta} \int_{\Omega} f(x, z) d\mathbb{P}_{\omega}(z) d\lambda_{\theta}(x)$$

defines a version of $E[f|\mathcal{B}_0 \otimes \mathcal{F}_0]$. □

Chapter 4

Quenched Asymptotics of Normalized Fourier Averages

In this chapter we will introduce the results on asymptotic distributions to be proved along this monograph. The main results are theorems 15.1, 16.3, 17.1 and 17.2 (theorems 16.5 and 16.6 can be seen as versions of the previous ones refined by the introduction of additional structure).

For some of the results, including the main ones, we will limit our discussion to the presentation of the statements and to the comments necessary to clarify their meaning. We will nonetheless provide proofs of some of the corollaries and “secondary” results whenever they can be reached in a straightforward manner from the discussions already made.

This chapter is organized as follows: in Section 15 we present the *Central Limit Theorem for Fourier transforms at (a.e.)fixed frequencies* (Theorem 15.1), which extends to the quenched setting Theorem 5.5 and opens the door to several questions regarding the validity of this quenched convergence in stronger forms.

Then, in Section 16, we address the first issue in the direction of these questions: the necessity of the “random” centering of the normalized ergodic averages in order to guarantee the conclusion of Theorem 15.1. We will state a general result (Theorem 16.3) showing that this is indeed a necessary condition, but we will still address, in Section 16.2, particular cases in which this normalization is irrelevant.

Finally, in Section 17, we will address the problem of extending the quenched central limit theorems under consideration to corresponding quenched invariance principles. We will state a result (Theorem 17.1) showing that this is indeed possible in the sense of *averaged frequencies*. The (stronger) version for fixed frequencies remains open, but a special case (Theorem 17.2), and some of its consequences, are discussed in Section 17.2.

Some sections have a part dedicated to “general comments”. The purpose of these discussions is to clarify the meaning of the results previously given, to describe some of the

relations between them, and to motivate the discussions that follow both in the corresponding as in further sections.

15 The quenched CLT for Fourier Transforms

The purpose of this section is to present the most general version of the quenched central limit theorem for Fourier Transforms available in this monograph. The result is the following.

Theorem 15.1 (The Quenched Central Limit Theorem for Fourier Transforms). *Let $(X_k)_{k \in \mathbb{Z}} = (T^k X_0)_{k \in \mathbb{Z}}$ be a square-integrable ergodic process (Definition 4.1) adapted to an increasing T -filtration $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ (Definition 4.2). Assume that \mathcal{F}_∞ (Definition 4.4) is countably generated (Definition 1.5), that $E_0 = E[\cdot | \mathcal{F}_0]$ is regular (Definition 11.2), denote by $S_n(\theta)$ the n -th discrete Fourier Transform of $(X_k)_{k \in \mathbb{Z}}$ (Definition 2.6) and let*

$$Y_n(\theta) := \frac{1}{\sqrt{n}}(S_n(\theta) - E_0 S_n(\theta)). \quad (4.1)$$

Then there exist $I \subset [0, 2\pi)$ with $\lambda(I) = 1$ such that the following holds:

1. For every $\theta \in I$, there exists a nonnegative number $\sigma(\theta)$ such that

$$\sigma^2(\theta) = \lim_n E_0 |Y_n(\theta)|^2, \quad \mathbb{P}\text{-a.s. and in } L^1_{\mathbb{P}}. \quad (4.2)$$

2. If N_1, N_2 denote independent standard normal random variables and $i := \sqrt{-1}$, then for every $\theta \in I$, the process $Y_n(\theta)$ converges in the quenched sense (Definition 11.1) with respect to \mathcal{F}_0 to

$$Y(\theta) = \frac{\sigma(\theta)}{\sqrt{2}}(N_1 + iN_2), \quad (4.3)$$

(or, what is the same, $Y_n(\theta)$ convergence in the quenched sense to a bivariate normal, centered variable with covariance matrix (1.80)).

In addition, $\theta \mapsto \sigma^2(\theta)$ is the spectral density (Definition 5.3) of the process $(X_k - E_{-\infty} X_k)_{k \in \mathbb{Z}}$, where $E_{-\infty}$ denotes the conditional expectation $E[\cdot | \mathcal{F}_{-\infty}]$ with respect to $\mathcal{F}_{-\infty}$.

Before moving on to further comments, let us state the following Corollary, whose proof is given in full detail to facilitate further discussions.

Corollary 15.2. *In the context of Theorem 15.1, and denoting by $Y_n : [0, 2\pi) \times \Omega \rightarrow \mathbb{C}$ and $Y : [0, 2\pi) \times \Omega' \rightarrow \mathbb{C}$ the functions defined respectively by $Y_n(\theta, \omega) = Y_n(\theta)(\omega)$ and $Y(\theta, \omega') = Y(\theta)(\omega')$, there exists a set $\Omega_0 \subset \Omega$ with $\mathbb{P}\Omega_0 = 1$ such that for every $\omega \in \Omega_0$, $Y_n \Rightarrow Y$ under $\lambda \times \mathbb{P}_\omega$.*

Proof: First note that if $\mathcal{B}_0 = \{\emptyset, [0, 2\pi)\}$ is the trivial sigma-algebra and $\lambda_\theta := \lambda$ for all $\theta \in [0, 2\pi)$ then, by Proposition 14.1, the family of measures

$$\{\lambda_\theta \times \mathbb{P}_\omega\}_{\omega \in \Omega}$$

is a decomposition of $E[\cdot | \mathcal{B}_0 \otimes \mathcal{F}_0]$.

Now, the functions Y_n, Y given in the statement of Corollary 15.2 are clearly measurable with respect to the respective product sigma algebras and therefore, in virtue of Proposition 14.1, we can read the statement of Theorem 15.1 in the following way: *for any continuous and bounded function $f : \mathbb{C} \rightarrow \mathbb{R}$*

$$E[f \circ Y_n | \mathcal{B} \otimes \mathcal{F}_0] \rightarrow_n E[f \circ Y(\theta, \cdot)] \quad \lambda \times \mathbb{P}\text{-a.s.} \quad (4.4)$$

Let us explain this in detail: note that by an application of Proposition 14.1, the function at the right-hand side in (4.4) is a version of the conditional expectation of $f \circ Y$ with respect to the sigma field $\mathcal{B} \otimes \{\emptyset, \mathcal{F}'\}$. Therefore this function is $\mathcal{B} \otimes \mathcal{F}'$ -measurable and, since it is constant over Ω' for θ fixed, it is \mathcal{B} -measurable. By regarding it as constant on Ω for θ fixed, it can be considered $\mathcal{B} \otimes \mathcal{F}$ -measurable. This shows that the set where the convergence in (4.4) occurs belongs to $\mathcal{B} \otimes \mathcal{F}$.

Now, a further application of Proposition 14.1 shows that

$$(\theta, \omega) \mapsto \int_{\Omega} f \circ Y_n(\theta, z) d\mathbb{P}_\omega(z)$$

defines a version of $E[f \circ Y_n | \mathcal{B} \otimes \mathcal{F}_0]$, and since for λ -a.e. fixed θ ,

$$\int_{\Omega} f \circ Y_n(\theta, z) d\mathbb{P}_\omega(z) \rightarrow E[f \circ Y(\theta, \cdot)],$$

\mathbb{P} -a.s., we deduce that the set where the convergence in (4.4) occurs has, indeed, product measure one.

It follows from Proposition 11.1 that Y_n converges to Y in the quenched sense with respect to $\mathcal{B}_0 \otimes \mathcal{F}_0$. The conclusion follows at once from the observation at the beginning of this proof and Proposition 13.1. \square

General Comments

Note that the convergence in (4.4) resembles the convergence that follows from Theorem 5.6 by evaluating the corresponding random functions at $t = 1$. As we shall see, Corollary 15.2 can indeed be extended to a quenched invariance principle without imposing any further hypothesis to the processes under consideration (see Theorem 17.1 below). At the moment of writing this monograph this is not the case for Theorem 15.1, whose extension to an invariance principle will be possible for us only at the expense of further assumptions.

With regards to the statement of Theorem 15.1, the following comments are worth at this point.

1. First, note that Theorem 15.1 is apparently a re-statement of Theorem 5.5: it basically emerges from that result by replacing “ X_k ” by “ $X_k - E_0 X_k$ ” and “convergence in distribution” by “quenched convergence”. Note nevertheless that the process $(X_k - E_0 X_k)_{k \in \mathbb{Z}}$ is generally non-stationary (all its entries are zero for $k \leq 0$), and therefore that substitution brings us outside of the hypotheses of Theorem 5.5.
2. Another one of the hypotheses of Theorem 5.5 is missing from the statement of Theorem 15.1: the regularity of $(X_k)_{k \in \mathbb{Z}}$ (Definition 5.4), but this actually “can be obtained” from the theory already developed via a simple substitution, as we proceed now to explain.

First, the process

$$(X_{-\infty, k})_{k \in \mathbb{Z}} := (X_k - E_{-\infty} X_k)_{k \in \mathbb{Z}}$$

is stationary and regular. Indeed: $(X_{-\infty, k})_{k \in \mathbb{Z}}$ is stationary by (1.59), and an application of Proposition 5.3 (which is actually implicit in the statement of Theorem 15.1) shows that it is regular.

Now, since $E_0 E_{-\infty} = E_{-\infty}$, a simple computation shows that (see the notation in Definition 2.6)

$$S_n((X_k)_k, \theta, \cdot) - E_0 S_n((X_k)_k, \theta, \cdot) = S_n((X_{-\infty, k})_k, \theta, \cdot) - E_0 S_n((X_{-\infty, k})_k, \theta, \cdot) \quad (4.5)$$

and therefore we can study the asymptotics of $Y_n(\theta)$ assuming, via the substitution of X_k by $X_k - E_{-\infty} X_k$ for all $k \in \mathbb{Z}$, that $(X_k)_{k \in \mathbb{Z}}$ is stationary, centered, and regular.

3. Now consider the following observation: in the context of Theorem 15.1, the process

$$Z_n(\theta) := \frac{1}{\sqrt{n}} S_n(\theta)$$

satisfies

$$Z_n(\theta) = Y_n(\theta) + \frac{E_0 S_n(\theta)}{\sqrt{n}}, \quad (4.6)$$

and since $Y_n(\theta)$ converges in the quenched sense, and therefore in distribution to (4.3), we have the following corollary.

Corollary 15.3. *Under the hypotheses of Theorem 15.1, the conclusion of Theorem 5.5 remains true (without necessarily assuming the regularity of $(X_k)_{k \in \mathbb{Z}}$) if for λ -a.e θ ,*

$$\frac{E_0 S_n(\theta)}{\sqrt{n}} \Rightarrow_n 0, \quad (4.7)$$

in which case $\theta \mapsto \sigma^2(\theta)$ is the spectral density of the (regular) process $(X_k - E_{-\infty} X_k)_{k \in \mathbb{Z}}$ (where “ $E_{-\infty}$ ” is as in the last statement of Theorem 15.1). In particular, the conclusion of Theorem 5.5 follows under the hypotheses of Theorem 15.1 if $(X_k)_{k \in \mathbb{Z}}$ is regular.

Proof: Only the last statement requires a proof. To do so we will prove that, under the hypothesis of Theorem 15.1, the hypothesis of regularity in Theorem 5.5 imply the fulfillment of (4.7).

Using the notation introduced in theorems 5.5 and 15.1 we have by orthogonality that, for every $\theta \in I$

$$E|Z_n(\theta)|^2 = E|Z_n(\theta) - E_0 Z_n(\theta)|^2 + E|E_0 Z_n(\theta)|^2 = E[E_0|Y_n(\theta)|^2] + E|E_0 Z_n(\theta)|^2$$

and it follows from Theorem 5.4, Theorem 15.1, and Fatou's lemma that for λ -a.e θ

$$\sigma^2(\theta) = \limsup_n E|Z_n(\theta)|^2 \geq \liminf_n E[E_0|Y_n(\theta)|^2] + \limsup_n E|E_0 Z_n(\theta)|^2 \geq$$

$$E[\liminf_n E_0|Y_n(\theta)|^2] + \limsup_n E|E_0 Z_n(\theta)|^2 = \sigma^2(\theta) + \limsup_n E|E_0 Z_n(\theta)|^2,$$

which implies that $E_0 Z_n(\theta)$ converges to zero in $L^2_{\mathbb{P}}$. This clearly implies (4.7). \square

16 The Random Centering

This leaves us with a question about the “missing” element on the statement in Theorem 5.5: the random centering “ $-E_0 S_n(\theta)$ ” in the definition of $Y_n(\theta)$.

More precisely, consider the following observations: every process satisfying the hypotheses of Theorem 5.5 satisfies the hypotheses of Theorem 15.1, and by the arguments following the statement of Theorem 15.1, the processes involved in the statement of Theorem 15.1 can be assumed to satisfy the hypotheses of Theorem 5.5.

Even more, in Corollary 15.3 we obtained the convergence in distribution of the normalized discrete Fourier transforms

$$Z_n(\theta) := \frac{1}{\sqrt{n}} S_n(\theta) \tag{4.8}$$

by using the convergence in distribution of $Y_n(\theta)$ and the “ad hoc” hypothesis for the remainder, but the following question is still to be addressed.

Question: *can we actually prove that $(Z_n(\theta))_{n \in \mathbb{N}}$ converges in the quenched sense under the hypotheses of Theorem 5.5?*

16.1 Necessity of the Random Centering

To begin the discussion regarding the question above note that by (4.6), and since $Y_n(\theta)$ admits the same quenched limit as the limit (in distribution) of $Z_n(\theta)$, the “perturbation” to quenched convergence, if any, is due to the behavior of $E_0 S_n(\theta)/\sqrt{n}$ under \mathbb{P}_ω .

This can actually be described in a very precise way, as stated by the following theorem.

Theorem 16.1 (Possible Quenched Limits for the Non-centered Normalized Averages). *In the context of Theorem 15.1, given $\theta \in I$ and denoting by $Z_n(\theta)$ a (fixed) version of the random variable in (4.8) ($n \in \mathbb{N}$) and by $E_0 Z_n(\theta)$ a (fixed) version of $E[Z_n(\theta)|\mathcal{F}_0]$, there exists $\Omega_\theta \subset \Omega$ with $\mathbb{P}\Omega_\theta = 1$ such that, for $\omega \in \Omega_\theta$ the following are equivalent*

1. $Z_n(\theta)$ is convergent in distribution under \mathbb{P}_ω .

2. There exists

$$L_\theta(\omega) = \lim_n E_0[Z_n(\theta)](\omega), \quad (4.9)$$

and $Z_n(\theta) \Rightarrow Y(\theta) + L_\theta(\omega)$ under \mathbb{P}_ω .

The proof of this theorem is deferred to Section 23, but we will use it at this point to prove the following corollary.

Corollary 16.2. *In the context of Theorem 15.1, denoting by $Z_n(\theta)$ the random variable (4.8), and assuming that $(X_k)_{k \in \mathbb{Z}}$ is regular (Definition 5.4), the following are equivalent for $\theta \in I$.*

1. $Z_n(\theta)$ converges in the quenched sense as $n \rightarrow \infty$.
2. $E_0 Z_n(\theta) \rightarrow_n 0$, \mathbb{P} -a.s.,

in which case the (quenched) limit of $Z_n(\theta)$ is $Y(\theta)$.

Proof: Fix $\theta \in I$. Since, by Theorem 5.5, $Z_n(\theta) \Rightarrow Y(\theta)$ (under \mathbb{P}), the only possible quenched limit of $Z_n(\theta)$ is certainly $Y(\theta)$ (see the paragraph following Remark 11.1).

Now, by Proposition 13.1, the quenched convergence of $Z_n(\theta)$ to $Y(\theta)$ is equivalent to the following: there exists a set $\Omega_{\theta,1} \subset \Omega$ with $\mathbb{P}\Omega_{\theta,1} = 1$ such that for every $\omega \in \Omega_{\theta,1}$

$$Z_n(\theta) \Rightarrow Y(\theta)$$

under \mathbb{P}_ω as $n \rightarrow \infty$.

Now note that

$$Z_n(\theta) = Y_n(\theta) + E_0 Z_n(\theta). \quad (4.10)$$

By Proposition 13.1 and Theorem 15.1, there exists $\Omega_{\theta,2}$ with $\mathbb{P}\Omega_{\theta,2} = 1$ such that for every $\omega \in \Omega_{\theta,2}$

$$Y_n(\theta) \Rightarrow Y(\theta)$$

under \mathbb{P}_ω as $n \rightarrow \infty$. The conclusion follows considering

$$\omega \in \bigcap_{k=0}^2 \Omega_{\theta,k}$$

where $\Omega_{\theta,0}$ is the set specified in Theorem 16.1 and applying Proposition 8.3 (use the complex version of Proposition 8.1 (respectively, Proposition 8.2) when $\sigma(\theta) > 0$ (respectively, when $\sigma(\theta) = 0$)). \square

We return to the question above, that about the quenched convergence (in general) of $Z_n(\theta)$ for $\theta \in I$. The actual answer is *no*, as our next main result shows.

Theorem 16.3 (An Example of non-Quenched Convergence). *There exist \mathcal{F} , \mathcal{F}_0 , T , and $(X_k)_{k \in \mathbb{Z}}$ as in the hypotheses of Theorem 5.5 such that $E_0 := E[\cdot | \mathcal{F}_0]$ is regular and for any decomposition $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$ of E_0 (Definition 11.2)*

$$Z_n(\theta) = \frac{1}{\sqrt{n}} S_n(\theta)$$

admits no limit in distribution under \mathbb{P}_ω for every $\theta \in [0, 2\pi)$ and \mathbb{P} -a.e ω .

General Comments

With regards to the results in this section it is important to observe the following: for the process $(X_n)_{n \in \mathbb{Z}}$ to be constructed along the proof of Theorem 16.3, if $Y_n(\theta)$ is given by (4.1), $Y(\theta)$ is given by (4.3) and $Z_n(\theta)$ is given by (4.8), then certainly

$$Z_n(\theta) \Rightarrow Y(\theta)$$

as $n \rightarrow \infty$ for λ -almost every θ . Theorem 16.3 not only states that this convergence is *not* quenched, but it states that $Z_n(\theta)$ cannot converge when started at \mathbb{P} -a.e ω , this is, $Z_n(\theta)$ does not admit a limit (in distribution) under \mathbb{P}_ω for \mathbb{P} -a.e ω . As a matter of fact, we will see that for this process

$$\mathbb{P}[\limsup_n |E_0 Z_n(\theta)| = \infty] = 1, \quad (4.11)$$

which makes impossible the convergence under \mathbb{P}_ω for \mathbb{P} -almost every ω in virtue of Theorem 16.1.

This enforces the intuitive idea that \mathcal{F}_0 represents the “deterministic part” of the processes in question. Note again that, even if we can prove that (4.9) exists for \mathbb{P} -a.e ω , we *cannot* a priori conclude that $Z_n(\theta)$ converges in the quenched sense, because according to our definition of quenched convergence and Proposition 13.1, the asymptotic distribution of $Z_n(\theta)$ under \mathbb{P}_ω must be independent of ω . Of course, this is more a limitation of our definition of quenched convergence (Definition 11.1) than an inherent pathology of the behavior of a (\mathbb{P} -convergent) process under the measures \mathbb{P}_ω .

16.2 Cases of Quenched Convergence without Random Centering

Now consider the following observation: by the proof of Corollary 15.3 and (4.5), for every $\theta \in I$, $E_0(Z_n(\theta) - E_{-\infty} Z_n(\theta)) \rightarrow 0$ in $L^1_{\mathbb{P}}$. It follows from Fatou’s lemma (see the proof of Theorem 16.4 on [11]) that, if we assume the condition

$$\sup_n |E_0(Z_n(\theta) - E_{-\infty} Z_n(\theta))| \in L^1_{\mathbb{P}}, \quad (4.12)$$

then

$$E[\limsup_n |E_0(Z_n(\theta) - E_{-\infty} Z_n(\theta))|] \leq \limsup_n E[|E_0(Z_n(\theta) - E_{-\infty} Z_n(\theta))|] = 0,$$

which is possible if and only if $E_0(Z_n(\theta) - E_{-\infty} Z_n(\theta)) \rightarrow 0$, \mathbb{P} -a.s. Thus the following result follows from Corollary 16.2.

Corollary 16.4. *In the context of Theorem 15.1, denote by $Z_n(\theta)$ the random variable given in (4.8). Then the validity of condition (4.12) for $\theta \in I$ implies that $(Z_k(\theta) - E_{-\infty} Z_k(\theta))_{k \in \mathbb{N}}$ converges to $Y(\theta)$ in the quenched sense. In particular, the condition*

$$\sup_n |E_0 Z_n(\theta)| \in L^1_{\mathbb{P}} \quad (4.13)$$

for $\theta \in I$ implies that $(Z_k(\theta))_{k \in \mathbb{N}}$ converges to $Y(\theta)$ in the quenched sense if $(X_k)_{k \in \mathbb{Z}}$ is regular.

Let us give two more results regarding the quenched convergence of $(Z_n(\theta))_{n \in \mathbb{N}}$ for $\theta \in I$ in terms of decay of correlations, whose proof will be given in Section 26.

Theorem 16.5. *In the context of Theorem 15.1, denote by $Z_n(\theta)$ the random variable given in (4.8). If the condition*

$$\sum_{k \in \mathbb{N}^*} \frac{|E_0[X_k - X_{k-1}]|^2}{k} < \infty, \quad \mathbb{P}\text{-a.s.} \quad (4.14)$$

holds, there exists $J \subset I$ with $\lambda(J) = 1$ such that, for every $\theta \in J$, $Z_n(\theta) \Rightarrow_n Y(\theta)$ in the quenched sense.

Our last theorem in this direction is related to the *Maxwell and Woodroffe condition*, and its proof is essentially an application of results found by Cuny and Merlevéde in [18]. The statement is the following.

Theorem 16.6 (Quenched Convergence under the Maxwell-Woodroffe Condition). *In the context of Theorem 15.1, and given $\theta \in I$, denote by $Z_n(\theta)$ the random variable given in (4.8), then the Maxwell and Woodroffe condition*

$$\sum_{k \in \mathbb{N}^*} \frac{\|E_0 S_k(\theta)\|_{\mathbb{P},2}}{k^{3/2}} < \infty \quad (4.15)$$

implies the quenched convergence of $Z_n(\theta)$ to $Y(\theta)$.

Remark 16.1. It is possible to relax the assumption “ $\theta \in I$ ” to “ $e^{2i\theta} \notin \text{Spec}_p(T)$ ” in the hypotheses of Theorem 16.6 by using a direct martingale approximation also presented in [18]. See the proof of Theorem 6 in [5] for details.

17 Quenched Functional Central Limit Theorem

Finally, let us address the question of the validity of the quenched Central Limit Theorem in its functional form.

To begin with, let us recall the definition of the space (S, d) of complex valued cadlag functions on $[0, \infty)$: Definition 7.1), and that a random element of S is (by definition) a measurable function $W : \Omega' \rightarrow S$ where $(\Omega', \mathcal{F}', \mathbb{P}')$ is a probability space and S is endowed with its Borel sigma algebra \mathcal{S} . By an adaptation of the theory for $D[[0, \infty)]$ (see for instance Theorem 16.6 in [10]), \mathcal{S} is also the sigma algebra generated by the finite dimensional cylinders

$$H_{t_1 \dots t_k, A} := [\pi_{t_1 \dots t_k} \in A], \quad (4.16)$$

where A is a Borel set in \mathbb{C}^k , $0 \leq t_1 \leq \dots \leq t_k$, and $\pi_{t_1 \dots t_k}$ is given by (2.10).

It follows (see the argument in [10], p.84) that if $(\Omega', \mathcal{F}', \mathbb{P}')$ is a probability space, $W : \Omega' \rightarrow S$ is a random element of S if and only if for every $t \geq 0$, $\pi_t \circ W$ (i.e., the function $\omega' \mapsto W(\omega')(t)$) is a random variable in $(\Omega', \mathcal{F}', \mathbb{P}')$.

The Question

Here the problem is the following: consider the setting in the hypothesis of Theorem 15.1, and for $(\theta, \omega) \in [0, 2\pi) \times \Omega$, consider the function $W_n : [0, \infty) \times \Omega \rightarrow \mathbb{C}$ given by

$$W_n(\theta, \omega)(t) := \frac{S_{[nt]}(\theta, \omega) - E_0[S_{[nt]}(\theta, \cdot)](\omega)}{\sqrt{n}}. \quad (4.17)$$

This is: for fixed θ, ω and n , $W_n(\theta, \omega)$ takes the value

$$\frac{S_k(\theta, \omega) - E_0[S_k(\theta, \cdot)](\omega)}{\sqrt{n}}$$

whenever $t \in [k/n, (k+1)/n)$.

Note that for fixed (θ, ω) , $W_n(\theta, \omega)$ is an element of S , and that there are two ways in which we can regard W_n as a random element of S :

1. *Fixed frequency approach.* For fixed $\theta \in [0, 2\pi)$, consider the function $W_n(\theta) : \Omega \rightarrow S$

$$W_n(\theta)(\omega) = W_n(\theta, \omega). \quad (4.18)$$

Then $W_n(\theta)$ is a random element of S .

2. *Averaged frequency approach.* Consider the product space $([0, 2\pi) \times \Omega, \mathcal{B} \otimes \mathcal{F}, \lambda \times \mathbb{P})$. Then the function $W_n : [0, 2\pi) \times \Omega \rightarrow S$ is a random element of S .

Our goal is to give results on the quenched convergence of W_n from both the fixed frequency and the averaged frequency points of view. Note that, by the discussion in Section 9 (see the discussion following Theorem 17.1), results for λ -almost every fixed frequency imply results for averaged frequencies.

17.1 The Invariance Principle for Averaged Frequencies

Our first result concerns the validity of the quenched Invariance Principle under the averaged frequency approach. It is the following:

Theorem 17.1 (The Quenched Invariance Principle for Averaged Frequencies). *In the setting of Theorem 15.1, let B_1, B_2 be independent standard Brownian motions on $[0, \infty)$ defined on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. Consider the trivial sigma-algebra $\mathcal{B}_0 := \{\emptyset, [0, 2\pi)\} \subset \mathcal{B}$, and let S be the space of cadlag complex valued functions with the Skorohod distance (Definition 7.1). Then the sequence $(W_n)_{n \in \mathbb{N}^*}$ of random elements of S specified by (4.17) converges in the quenched sense with respect to $\mathcal{B}_0 \otimes \mathcal{F}_0$ to the random function $B : [0, 2\pi) \times \Omega' \rightarrow S$ specified by*

$$B(\theta, \omega') = \frac{\sigma(\theta)}{\sqrt{2}}(B_1(\omega') + iB_2(\omega')). \quad (4.19)$$

Equivalently, for any decomposition $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$ of E_0 (Definition 11.2), there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}\Omega_0 = 1$ such that for every $\omega \in \Omega_0$

$$W_n \Rightarrow B \quad \text{under } \lambda \times \mathbb{P}_\omega. \quad (4.20)$$

This theorem should be compared with Theorem 5.6: it plays a role with respect to this theorem similar to that of Theorem 15.1 with respect to Theorem 5.5.

17.2 Invariance Principles for Almost Every Fixed Frequencies

Of course, we would like to give an extension of Theorem 15.1 in the direction of an invariance principle valid for λ -a.e fixed frequency, which in particular would imply the convergence stated in Theorem 17.1.

To be more precise, note that if we are able to prove that for λ -almost every fixed θ the sequence $(W_n(\theta))_{n \geq 0}$ of random elements of S (defined on $(\Omega, \mathcal{F}, \mathbb{P})$) converges in the quenched sense to $B(\theta, \cdot)$ with respect to \mathcal{F}_0 then, by an argument similar to that in the proof of Corollary 15.2, the quenched convergence of W_n with respect to $\mathcal{B}_0 \otimes \mathcal{F}_0$ follows at once.

The validity of the quenched invariance principle for λ -almost every θ is a problem under current research.¹ In this work, we will give a result in the direction of Hannan-like conditions guaranteeing its fulfillment.²

Motivation

In what follows, the notations and the assumptions are those given in Theorem 15.1.

To illustrate our last results we start by considering the *Hannan* condition: recall the definition (1.65) of the projection operators \mathcal{P}_k ($k \in \mathbb{Z}$). We say that $(X_k)_{k \in \mathbb{Z}}$ satisfy the *Hannan Condition* if

$$\sum_{n \in \mathbb{N}} \|\mathcal{P}_0 X_n\|_2 < \infty. \quad (4.21)$$

Cuny and Volný showed, in [21], that in the context of Theorem 17.1, condition (4.21) guarantees that $W_n(0)$ converges to

$$B'(0) = \sigma(0)B_1.$$

where

$$\sigma^2(0) = \lim_n E_0[|Y_n(0)|^2]$$

\mathbb{P} -a.s. (see the notation in Theorem 15.1).

In spite of the fact that this is a quenched result for (only) one frequency, and that the quenched asymptotic distribution of $W_n(\theta, \cdot)$ does not correspond to a two-dimensional Brownian motion (but to a one-dimensional one), we will see that this condition is actually strong enough to guarantee the quenched convergence of $W_n(\theta, \cdot)$ at every $\theta \neq 0$ provided that $e^{2i\theta} \notin \text{Spec}_p(T)$.

¹At the moment of writing this monograph, the author ignores whether this stronger form of the invariance principle can be proved without assumptions additional to those in Theorem 15.1.

²But other approaches are possible. For instance via the results in [18] (see the proof of Theorem 16.6 for an illustration of the use of these results).

Our main result in this direction depends on the following condition

$$\sum_{n \geq 0} \|\mathcal{P}_0(X_{n+1} - X_n)\|_2 < \infty, \quad (4.22)$$

which is clearly a “weak” version of the Hannan condition³ (4.21). The result is the following.

Theorem 17.2 (A Quenched Invariance Principle for Fixed Frequencies). *With the notation and assumptions of Theorem 17.1, and assuming (4.22), if $e^{2i\theta} \notin \text{Spec}_p(T)$, then $W_n(\theta, \cdot)$ converges in the quenched sense to*

$$\omega' \mapsto \frac{\sigma(\theta)}{\sqrt{2}} (B_1(\omega') + iB_2(\omega')). \quad (4.23)$$

where $\sigma(\theta)$ is given as in Theorem 15.1 (see (4.2)).

It is worth to further specify a case in which the set of frequencies where the asymptotic distribution is as in (4.23) can be easily described. To motivate the following Theorem recall that T is weakly mixing if and only if $\text{Spec}_p(T) = \{1\}$ (see [42], Section 8 for a review of this and other related facts).

Now, as a subgroup of \mathbb{T} , $\text{Spec}_p(T)$ is finite (actually: closed) if and only if there exists $m \in \mathbb{N}^*$ such that

$$\text{Spec}_p(T) := \{e^{2\pi ki/m}\}_{k=0}^{m-1}. \quad (4.24)$$

In other words $\text{Spec}_p(T)$ is finite if and only if it consists of the points in the unit circle given by the rational rotations by an angle of $2\pi/m$ or, what is the same, by the m -th roots of unity.

Our last result is the following.

Corollary 17.3. *Assume that $\text{Spec}_p(T)$ is finite and its elements are the m -th roots of unity. Under the hypothesis and the notation in Theorem 17.2, $W_n(\theta)$ converges in the quenched sense to (4.23) for all $\theta \in [0, 2\pi)$ such that $e^{2im\theta} \neq 1$. If T is in particular weakly mixing, (4.23) describes the asymptotic quenched limit of $W_n(\theta)$ for all $\theta \neq 0, \pi$.*

Proof: Immediate from (4.24) and Theorem 17.2. \square

³To see that this condition is strictly weaker than the Hannan condition consider the process

$$X_k := \sum_{j \geq 1} \frac{1}{j} x_{k-j}$$

where $(x_j)_{j \in \mathbb{Z}}$ are the coordinate functions in $\mathbb{R}^{\mathbb{Z}}$, seen as an i.i.d sequence in L^2 , T is the left shift, and $\mathcal{F}_0 = \sigma(x_k)_{k \leq 0}$ (see Example 1 in page 29).

Part II

Proofs

General Setting

In addition to the notation introduced at the beginning, the following setting will be fixed throughout this part of the monograph: $(\Omega, \mathcal{F}, \mathbb{P})$ will be a fixed probability space. $T : \Omega \rightarrow \Omega$ will be a fixed invertible, bimeasurable measure-preserving transformation (Definition 1.1). As before, T will (also) denote the Koopman operator associated to the map T (Definition 1.2), and $\text{Spec}_p(T)$ will denote its point spectrum (Definition 1.3). $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ will be a fixed T -filtration (Definition 4.2) where \mathcal{F}_0 is countably generated (Definition 1.5), and given $k \in \mathbb{Z} \cup \{-\infty, \infty\}$, we will denote by E_k the conditional expectation with respect to \mathcal{F}_k , where $\mathcal{F}_{\pm\infty}$ are given via Definition 4.4.

We will assume that $E_0 := E[\cdot | \mathcal{F}_0]$ is regular, $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$ will be a fixed decomposition of E_0 (Definition 11.2), and for a given $\omega \in \Omega$, E^ω will denote integration with respect to \mathbb{P}_ω . Given $p > 0$, we will also denote by $Id : L^p_{\mathbb{P}} \rightarrow L^p_{\mathbb{P}}$ the identity function (the domain of Id will be clear from the context). When needed, we will use explicitly the version of E_0 given by integration with respect to \mathbb{P}^ω : $E_0 X(\omega) := E^\omega X$ for every $X \in L^1_{\mathbb{P}}$. Such restriction will not be assumed without explicit indication.

Finally, B_1, B_2 will denote independent standard Brownian motions defined on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, and $N_j = B_j(1)$ ($j = 1, 2$) denote independent standard normal random variables on $(\Omega', \mathcal{F}', \mathbb{P}')$.

Dot Product

In what follows, we will use the notation $a \cdot b$ to denote the dot product between vectors in \mathbb{R}^n ($n \in \mathbb{N}^*$). Thus if $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are elements of \mathbb{R}^n

$$a \cdot b := a_1 b_1 + \dots + a_n b_n.$$

In particular, if $z = z_1 + iz_2$ and $w = w_1 + iw_2$ are complex numbers (with respective real and imaginary parts z_1, w_1 and z_2, w_2)

$$z \cdot w := z_1 w_1 + z_2 w_2.$$

Structure of the Arguments

Our goal in this part of the monograph is to prove the results stated, but not proved, in Chapter 4. The general structure of the forthcoming arguments is the following.

1. *Martingale case.* We will start by addressing the martingale case. More precisely, we will prove that if $\theta \in [0, 2\pi)$ is such that $e^{2i\theta} \notin \text{Spec}_p(T)$ (see Definition 1.3) and $D_0(\theta) \in L^2_{\mathbb{P}}(\mathcal{F}_0) \ominus L^2_{\mathbb{P}}(\mathcal{F}_{-1})$, then the conclusion no. 1. in Theorem 17.2 holds replacing X_0 by $D_0(\theta)$ in the statement of this theorem. In this case $\sigma^2(\theta) = E|D_0(\theta)|^2$.
2. *Martingale approximations, proof of Theorem 15.1.* The next step is the following: given a stationary process $(X_k)_k = (T^k X_0)_k$ with $X_0 \in L^2_{\mathbb{P}}$, we will construct a random element $D_0 : [0, 2\pi) \times \Omega \rightarrow L^2_{\mathbb{P}}$ with the property that for λ -a.e θ , $D_0(\theta, \cdot) \in L^2_{\mathbb{P}}(\mathcal{F}_0) \ominus L^2_{\mathbb{P}}(\mathcal{F}_{-1})$. We will then prove Theorem 15.1 by showing that for λ -a.e θ and \mathbb{P} -a.e ω

$$\|(Id - E_0)S_n((X_k)_k, \theta) - S_n((T^k D_0(\theta, \cdot))_k, \theta)\|_{\mathbb{P}_{\omega}, 2} = o(\sqrt{n}) \quad (4.25)$$

(see Definition 2.6 for the notation) and then applying Theorem 10.1 together with the martingale case and the discussions made before.

3. *Proof of theorems 17.1 and 17.2.* To achieve the proof of these two theorems we will first show that for \mathbb{P} -a.e ω

$$\left\| \max_{1 \leq k \leq n} |(Id - E_0)S_n((X_k(z))_k, \theta) - S_n((D_k(\theta, z))_k, \theta)| \right\|_{\lambda \otimes \mathbb{P}_{\omega}, 2} = o(\sqrt{n}) \quad (4.26)$$

where $D_k(\theta, z) := D_0(\theta, T^k z)$ (this map will be $\mathcal{B} \otimes \mathcal{F}_{\infty}$ -measurable). This will give the proof of Theorem 17.1 by an approximation argument again and the martingale results in Section 19.

Then we will see that, under the conditions in the hypothesis of Theorem 17.2, (4.26) holds for λ -a.e $\theta \in (0, 2\pi)$ fixed (actually, for every θ with $e^{2i\theta} \notin \text{Spec}_p(T)$) replacing $\lambda \otimes \mathbb{P}_{\omega}$ by \mathbb{P}_{ω} , which again implies the functional form of Theorem 15.1 by the martingale version previously proved.

4. *Proof of Theorem 16.3.* We will then prove Theorem 16.3 by specializing our study to the case explained in Example 1: we will see that there exist a sequence $(a_k)_{k \in \mathbb{Z}} \in l^2(\mathbb{N})$ generating a linear process with the property announced in Theorem 16.3.
5. *Proof of theorems 16.5 and 16.6.* The proofs of these results end the content of this monograph. We will achieve them by using the characterization of quenched convergence without random centering given in Corollary 16.2 (which is proved in previous sections), together with suitable interpretations of results present in the existing literature applied to the processes under our consideration.

Chapter 5

Martingale Case

This chapter is devoted to present the martingale theorems (Theorem 19.1 and Corollary 19.2) which will be used to prove the results on quenched asymptotics presented in Chapter 4 via suitable martingale approximations and transport theorems.

In Section 18, we introduce some results from the existing literature which will allow us to carry out the proof of Theorem 19.1 by specializing to the case under our consideration. Section 19 presents the aforementioned proofs of the martingale case.

18 Preliminary Results

In this short section we present some preliminary facts needed to prove Theorem 19.1 below, from which all the proofs of the (positive) results announced in Chapter 4 will follow via suitable martingale approximations. With the exception of Lemma 5 (proved first by Cuny et.al in [19]), the results presented here pertain to the classical literature, but we decided to include their statements due to their very specific role among the proofs of our main theorems. The setting is that explained in page 89.

Our first result is a lemma that will allows us, among other things, to characterize the asymptotic finite-dimensional distributions of the normalized discrete Fourier transforms of a martingale at a frequency not associated to an element of $\text{Spec}_p(T)$ (in the sense just to be stated).

Lemma 5. *Let $\theta \in [0, 2\pi)$ be such that $e^{-2i\theta} \notin \text{Spec}_p(T)$, let $p \geq 1$ and let $Y \in L^p_{\mathbb{P}}$. Then for every $z \in \mathbb{C}$*

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} E_{k-1}(z \cdot (T^k Y e^{ik\theta}))^2 = \frac{|z|^2}{2} E|Y|^2 \quad \mathbb{P}\text{-a.s. and in } L^p_{\mathbb{P}}, \quad (5.1)$$

where the (probability one) set Ω_θ of pointwise convergence does not depend on z .

Proof:¹ Let $z = z_1 + iz_2$, and note first that

$$E_{k-1}(z \cdot (T^k Y e^{ik\theta}))^2 = T^k E_{-1}(z \cdot (Y e^{ik\theta}))^2. \quad (5.2)$$

Now, using Euler's formula and the double-angle identities, it is an elementary (though somewhat tedious) exercise in trigonometry to prove that, if $z = z_1 + iz_2$ and $Y = Y_1 + iY_2$ (where $z_j, Y_j, j = 1, 2$ are real-valued) and $k \in \mathbb{N}$,

$$\begin{aligned} (z \cdot (Y e^{ik\theta}))^2 &= \left(\frac{z_1^2 + z_2^2}{2} (Y_1^2 + Y_2^2) \right) + ((z_2^2 - z_1^2) Y_1 Y_2 - (Y_2^2 - Y_1^2) z_1 z_2) \sin(2k\theta) + \\ &\quad ((z_1 Y_1 + z_2 Y_2)^2 - (Y_2 z_1 - Y_1 z_2)^2) \frac{\cos(2k\theta)}{2}, \end{aligned} \quad (5.3)$$

thus there exist real constants (depending on z) a_j, b_j ($j = 1, 2, 3$) such that

$$\begin{aligned} (z \cdot (Y e^{ik\theta}))^2 &= \frac{|z|^2}{2} |Y|^2 + \\ &\quad (a_1 Y_1^2 + a_2 Y_2^2 + a_3 Y_1 Y_2) \cos(2k\theta) + (b_1 Y_1^2 + b_2 Y_2^2 + b_3 Y_1 Y_2) \sin(2k\theta). \end{aligned} \quad (5.4)$$

The conclusion follows at once from (5.2), (5.4), Theorem 3.2 and Corollary 3.3, by taking Ω_θ as the set of probability one where, according to the notation on Theorem 3.2

$$S_n(E_{-1}|Y|^2, 0)/n \rightarrow E|Y|^2, \quad S_n(E_{-1}[Y_1 Y_2], 2\theta)/n \rightarrow 0, \quad \text{and} \quad S_n(E_{-1}|Y|^2, 2\theta)/n \rightarrow 0$$

as $n \rightarrow \infty$. □

The next two theorems are very classical. We will use them to prove our martingale limit theorems in the setting of discrete Fourier transforms in the quenched sense.

Theorem 18.1 (The Lindeberg-Lévy Theorem for Martingales). *For each $n \in \mathbb{N}^*$, let $\Delta_{n1}, \dots, \Delta_{nk}, \dots$ be a sequence of real-valued martingale differences with respect to some increasing filtration $\mathcal{F}_0^n \subset \dots \subset \mathcal{F}_k^n \subset \dots$. Define, for $1 \leq k \leq n$, $\sigma_{nk} := E[\Delta_{nk}^2 | \mathcal{F}_n^{k-1}]$. If for some $\sigma \geq 0$ the following two conditions hold*

1. $\sum_{k \geq 0} \sigma_{nk}^2 \Rightarrow \sigma^2$ as $n \rightarrow \infty$,
2. $\sum_{k \geq 0} E[\Delta_{nk}^2 I_{[\Delta_{nk} \geq \epsilon]}] \rightarrow 0$ as $n \rightarrow \infty$,

then $Z_n := \sum_{k \geq 0} \Delta_{nk} \Rightarrow \sigma N$ where N is a standard normal random variable.

Proof: [11], p.476. □

Theorem 18.2 (The Functional form of Theorem 18.1). *For each $n \in \mathbb{N}^*$, let $\Delta_{n1}, \dots, \Delta_{nk}, \dots$ be a sequence of real-valued martingale differences with respect to some increasing filtration $\mathcal{F}_0^n \subset \dots \subset \mathcal{F}_k^n \subset \dots$ and defines, for $1 \leq k \leq n$, $\sigma_{nk}^2 := E[\Delta_{nk}^2 | \mathcal{F}_n^{k-1}]$. If for some $\sigma \geq 0$ the following two conditions hold for every $t \geq 0, \epsilon > 0$*

1. $\sum_{k \leq nt} \sigma_{nk}^2 \Rightarrow_n \sigma^2 t$,

¹For an alternative explanation of this proof see the proof of relation (16) in [19].

$$2. \sum_{k \leq nt} E[\Delta_{nk}^2 I_{[\Delta_{nk} \geq \epsilon]}] \rightarrow_n 0,$$

then the random functions $X_n(t) := \sum_{k \leq nt} \Delta_{nk}$ converge in distribution to σW in the sense of $D[[0, \infty)]$, where W is a standard Brownian motion.

Proof: This is a slight reformulation of Theorem 18.2 in [10], (pp. 194-195): the case $\sigma > 0$ follows by a simple renormalization, and to cover the case $\sigma = 0$, note that the convergence (18.6) in [10] becomes a simple consequence of the definition given there of ζ_{nk} and the hypothesis (corresponding to $\sigma = 0$)

$$\sum_{k \leq nt} \sigma_{nk}^2 \Rightarrow 0$$

for every $t \geq 0$. □

19 Martingale Case

As already mentioned, all of the positive results in Section 4 follow from the following theorem via suitable martingale approximations.

Theorem 19.1 (The Quenched Invariance Principle for the Discrete Fourier Transforms of a Martingale). *Under the setting introduced in page 89, and given $\theta \in [0, 2\pi)$ such that $e^{-2i\theta} \notin \text{Spec}_p(T)$ (Definition 1.3), assume that $D_0(\theta) \in L_{\mathbb{P}}^2(\mathcal{F}_0) \ominus L_{\mathbb{P}}^2(\mathcal{F}_{-1})$ is given, and define the $(\mathcal{F}_{k-1})_{k \in \mathbb{N}^*}$ -adapted martingale $(M_k(\theta))_{k \in \mathbb{N}}$ by*

$$M_n(\theta) := \sum_{k=0}^{n-1} T^k D_0(\theta) e^{ik\theta} \quad (5.5)$$

for all $n \in \mathbb{N}$. Then the sequence $(V_k(\theta))_{k \in \mathbb{N}^*}$ of random elements of $D[[0, \infty), \mathbb{C}]$ defined by

$$V_n(\theta)(t) := M_{\lfloor nt \rfloor}(\theta) / \sqrt{n} \quad (5.6)$$

for every $n \in \mathbb{N}^*$, converges in the quenched sense with respect to \mathcal{F}_0 to the random function $B(\theta) : \Omega' \rightarrow D[[0, \infty), \mathbb{C}]$ given by

$$B(\theta)(\omega') = [E|D_0(\theta)|^2/2]^{1/2} (B_1(\omega') + iB_2(\omega')). \quad (5.7)$$

Remark 19.1. Before proceeding to the proof it is worth noticing the following: the conclusion of Theorem 15.1, specialized to this case, is a statement about the asymptotic distribution of the random variables $V_n(\theta)(1)$. Now, by Corollary 4.2 and the orthogonality under E_0 of $(T^k D_0(\theta))_{k \in \mathbb{N}}$,²

$$E[|D_0(\theta)|^2] = \lim_n \frac{1}{n} \sum_{k=1}^{n-1} E_0 T^k |D_0(\theta)|^2 = \lim_n \frac{1}{n} E_0 |M_n(\theta) - E_0 M_n(\theta)|^2$$

²Note that if $(k, r) \in \mathbb{N} \times \mathbb{N}^*$ is given then, since $T^r D_0(\theta) \in L_{\mathbb{P}}^2(\mathcal{F}_r) \ominus L_{\mathbb{P}}^2(\mathcal{F}_{r-1})$,

$$E_0[T^k D_0(\theta) T^{k+r} \overline{D_0(\theta)}] = E_0[T^k D_0(\theta) E_k T^{k+r} \overline{D_0(\theta)}] = E_0 T^k [D_0(\theta) E_0 T^r \overline{D_0(\theta)}] = 0.$$

so that the equality (4.2) is certainly verified in this case.

Proof of Theorem 19.1: Let us start by sketching the argument of the proof: we will see that there exists $\Omega_\theta \subset \Omega$ with $\mathbb{P}\Omega_\theta = 1$ such that for every $\omega \in \Omega_\theta$ the following holds:

- a. *The sequence of random functions $(V_n(\theta))_n$ in $D[[0, \infty), \mathbb{C}]$ is tight with respect to \mathbb{P}_ω . To prove this, we will actually prove the convergence in distribution of both the real and imaginary parts of $(V_n(\theta))_n$ to a Brownian motion via Theorem 18.2 (see the “Criteria for Tightness” in section 7.2).*
- b. *The finite dimensional asymptotic distributions under \mathbb{P}_ω of $(V_n(\theta))_n$ converge to those of two independent Brownian motions with the scaling $E[(D_0(\theta))^2]^{1/2}/\sqrt{2}$ under \mathbb{P}_ω . For this we will proceed via the Cramer-Wold theorem, using some of the results already presented.*

We go now to the details: first, we will assume, making it explicit only when necessary, that E_0 is the version of $E[\cdot | \mathcal{F}_0]$ given by integration with respect to the decomposing probability measures $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$ (see Definition 11.2).

Now denote, for every $k \in \mathbb{N}$

$$D_k(\theta) := T^k D_0(\theta). \quad (5.8)$$

Let $\Omega'_{\theta,1}$ be the set of probability one guaranteed by Lemma 5 for the case $Y = D_0(\theta)$. By Remark 11.3, there exists a set $\Omega_{\theta,1}$ with $\mathbb{P}\Omega_{\theta,1} = 1$ such that for every $\omega \in \Omega_{\theta,1}$

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} E_{k-1}(z \cdot (D_k(\theta)e^{ik\theta}))^2 = \frac{|z|^2}{2} E|D_0(\theta)|^2$$

\mathbb{P}_ω -a.s. for all $z \in \mathbb{C}$.

For such ω 's the first hypothesis of Theorem 18.2 is verified by the triangular arrays $(\operatorname{Re}(M_k(\theta)/\sqrt{n}))_{1 \leq k \leq n}$ and $(\operatorname{Im}(M_k(\theta)/\sqrt{n}))_{1 \leq k \leq n}$ ($n \in \mathbb{N}^*$) with respect to \mathbb{P}_ω , because they arise from the particular choices $z = 1$ and $z = i$ respectively.

To verify the second hypothesis in Theorem 18.2 we start from the \mathbb{P} -a.s. inequality

$$\begin{aligned} E_0 \left[\frac{1}{n} \sum_{k=0}^{n-1} ((\operatorname{Re}(D_k(\theta)e^{ik\theta}))^2 I_{[|\operatorname{Re}(D_k(\theta)e^{ik\theta})| \geq \epsilon\sqrt{n}]} + (\operatorname{Im}(D_k(\theta)e^{ik\theta}))^2 I_{[|\operatorname{Im}(D_k(\theta)e^{ik\theta})| \geq \epsilon\sqrt{n}]}) \right] \leq \\ E_0 \left[\frac{1}{n} \sum_{k=0}^{n-1} |D_k(\theta)|^2 I_{[|D_k(\theta)| \geq \epsilon\sqrt{n}]} \right]. \end{aligned} \quad (5.9)$$

Now, given $\eta > 0$ there exists $N \geq 0$ such that $\mu_N := E[|D_0(\theta)|^2 I_{[|D_0(\theta)|^2 \geq \epsilon^2 N]}] < \eta$, and therefore

$$\begin{aligned} \limsup_n \frac{1}{n} \sum_{k=0}^{n-1} E_0 T^k[|D_0(\theta)|^2 I_{[|D_0(\theta)|^2 \geq \epsilon^2 n]}] \leq \\ \limsup_n \frac{1}{n} \sum_{k=0}^{n-1} E_0 T^k[|D_0(\theta)|^2 I_{[|D_0(\theta)|^2 \geq \epsilon^2 N]}] = \mu_N \leq \eta \end{aligned} \quad (5.10)$$

over a set $\Omega_{\theta,\epsilon,\eta}$ with $\mathbb{P}\Omega_{\theta,\epsilon,\eta} = 1$, where we made use of Corollary 4.2. Without loss of generality, (5.9) holds for all $\omega \in \Omega_{\theta,\epsilon,\eta}$.

Denote by Z_n^ϵ the random variable at the left-hand side of the inequality (5.9) and note that, if we define

$$\Omega_{\theta,2} = \bigcap_{\epsilon>0, \eta>0} \Omega_{\theta,\epsilon,\eta} \quad (5.11)$$

where the intersection runs over rational ϵ, η , then $\mathbb{P}\Omega_{\theta,2} = 1$, and for every $\epsilon > 0$ and every $\omega \in \Omega_{\theta,2}$

$$\lim_n Z_n^\epsilon(\omega) = 0.$$

or, what is the same, for all $\omega \in \Omega_{\theta,2}$

$$\frac{1}{n} \sum_{k=0}^{n-1} ((\operatorname{Re}(D_k(\theta)e^{ik\theta}))^2 I_{[|\operatorname{Re}(D_k(\theta)e^{ik\theta})| \geq \epsilon\sqrt{n}]} + (\operatorname{Im}(D_k(\theta)e^{ik\theta}))^2 I_{[|\operatorname{Im}(D_k(\theta)e^{ik\theta})| \geq \epsilon\sqrt{n}]})$$

goes to 0 in $L_{\mathbb{P}_\omega}^1$ as $n \rightarrow \infty$.

Thus, if $\Omega_{\theta,3}$ is a set of probability one such that $(\operatorname{Re}(M_k(\theta)))_{k \in \mathbb{N}^*}$ and $(\operatorname{Im}(M_k(\theta)))_{k \in \mathbb{N}^*}$ is a $(\mathcal{F}_{k-1})_{k \in \mathbb{N}^*}$ -adapted martingale in $L_{\mathbb{P}_\omega}^2$ for all $\omega \in \Omega_{\theta,3}$ (Corollary 11.2), the hypotheses 1. and 2. in Theorem 18.2 are verified for all ω in the set Ω_θ defined by

$$\Omega_\theta := \bigcap_{k=1}^3 \Omega_{\theta,k}. \quad (5.12)$$

Since $\mathbb{P}\Omega_\theta = 1$ this finishes the proof of **a**.

To prove **b**, we will show that for any given $n \in \mathbb{N}$, any $\omega \in \Omega_\theta$, and any $0 \leq t_1 \leq \dots \leq t_n$, the $\mathbb{C}^n = \mathbb{R}^{2n}$ -valued process

$$(V_n(\theta)(t_1), V_n(\theta)(t_2) - V_n(\theta)(t_1), \dots, V_n(\theta)(t_n) - V_n(\theta)(t_{n-1}))$$

has the same asymptotic distribution as

$$\mathbf{B}^\theta(t_1, \dots, t_n) :=$$

$$[E|D_0(\theta)|^2/2]^{1/2}(B_1(t_1), B_2(t_1), B_1(t_2) - B_1(t_1), B_2(t_2) - B_2(t_1), \dots, B_2(t_n) - B_2(t_{n-1}))$$

under \mathbb{P}_ω and therefore, by the mapping theorem ([10], Theorem 2.7), the finite dimensional asymptotic distributions of $V_n(\theta)$ under \mathbb{P}_ω and those of (5.7) under \mathbb{P}' are the same.

For simplicity we will assume $n = 2$. The argument generalizes easily to an arbitrary $n \in \mathbb{N}$.

Our goal is thus to prove that for all $\omega \in \Omega_\theta$ and all $0 \leq s \leq t$ the asymptotic distribution of

$$\mathbf{V}_n^\theta(s, t) := (V_n(\theta)(s), V_n(\theta)(t) - V_n(\theta)(s)) \quad (5.13)$$

(a $\mathbb{C}^2 = \mathbb{R}^4$ -valued process) is the same under \mathbb{P}_ω as that of

$$\mathbf{B}^\theta(s, t) := [E|D_0(\theta)|^2/2]^{1/2}(B_1(s), B_2(s), B_1(t) - B_1(s), B_2(t) - B_2(s)) \quad (5.14)$$

under \mathbb{P}' .

To prove the convergence in distribution of (5.13) to (5.14) we will use the Cramer-Wold theorem. This is, we will see that for any $\omega \in \Omega_\theta$, any $0 \leq s \leq t$, and any

$$\mathbf{u} = (a_1, a_2, b_1, b_2) \in \mathbb{R}^4 \quad (5.15)$$

the asymptotic distribution under \mathbb{P}_ω of the stochastic process $(U_n)_{n \in \mathbb{N}^*}$ defined by

$$U_n := \mathbf{u} \cdot \mathbf{V}_n^\theta(s, t) \quad (5.16)$$

is that of a normal random variable with variance

$$\sigma_{\mathbf{u}, s, t}^2(\theta) := \frac{E[|D_0(\theta)|^2]}{2}((a_1^2 + a_2^2)s + (b_1^2 + b_2^2)(t - s)). \quad (5.17)$$

To do so we will verify the hypotheses of Theorem 18.1. Fix \mathbf{u} as above and note that

$$U_n = \sum_{k=0}^{\lfloor ns \rfloor} \eta_{nk}(a_1, a_2) + \sum_{k=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \eta_{nk}(b_1, b_2)$$

where

$$\eta_{nk}(x_1, x_2) = \frac{1}{\sqrt{n}}(x_1, x_2) \cdot e^{ik\theta} T^k D_0(\theta). \quad (5.18)$$

By the construction of Ω_θ , for every $0 \leq r$, every x_1, x_2 and every $\omega \in \Omega_\theta$, $(\eta_{nk}(x_1, x_2))_{0 \leq k \leq \lfloor nr \rfloor}$ is a triangular array of $(\mathcal{F}_k)_k$ -adapted (real-valued) martingale differences under \mathbb{P}_ω , and by Lemma 5 combined with Remark 11.3 we can assume that

$$\sum_{k \leq ns} E_{k-1}[\eta_{nk}^2(a_1, a_2)] + \sum_{ns < k \leq nt} E_{k-1}[\eta_{nk}^2(b_1, b_2)] \rightarrow_n \sigma_{\mathbf{u}, s, t}^2(\theta) \quad (5.19)$$

\mathbb{P}_ω -a.s.³ This verifies the first hypothesis in Theorem 18.1 under \mathbb{P}_ω for all $\omega \in \Omega_\theta$ for the triangular array defining U_n .

It remains to prove that if $\omega \in \Omega_\theta$ then

$$\sum_{k \leq ns} E_0[\eta_{nk}^2(a_1, a_2) I_{[|\eta_{nk}(a_1, a_2)| > \epsilon]}](\omega) \rightarrow 0. \quad (5.20)$$

This is, that for all $\omega \in \Omega_\theta$

$$\sum_{k \leq ns} \eta_{nk}^2(a_1, a_2) I_{[|\eta_{nk}(a_1, a_2)| > \epsilon]} \rightarrow 0$$

³More precisely: redefine Ω_θ above by intersecting it with the set Ω'_θ of elements ω for which the convergence in Lemma 5 happens \mathbb{P}_ω -a.s.

in $L^1_{\mathbb{P}_\omega}$.

To do so we depart from the Cauchy-Schwartz inequality to get that

$$\eta_{nk}^2(x_1, x_2) \leq \frac{1}{n}(x_1^2 + x_2^2)T^k|D_0(\theta)|^2,$$

so that the sum in (5.20) is bounded by

$$\frac{1}{n} \sum_{k \leq ns} E_0 T^k [(a_1^2 + a_2^2)|D_0(\theta)|^2 I_{[(a_1^2 + a_2^2)|D_0(\theta)|^2 \geq \epsilon^2 n]}].$$

This obviously goes to zero when $a_1 = a_2 = 0$. Otherwise it is the same as

$$(a_1^2 + a_2^2) \frac{1}{n} \sum_{k \leq ns} E_0 T^k [|D_0(\theta)|^2 I_{[|D_0(\theta)|^2 \geq \epsilon^2 n / (a_1^2 + a_2^2)}],$$

which, again, goes to zero as $n \rightarrow \infty$ for every $\omega \in \Omega_\theta$. \square

Remark 19.2. When necessary, *specially when discussing quenched convergence in the product space* $([0, 2\pi) \times \Omega, \mathcal{B} \otimes \mathcal{F})$, we will specify the dependence on $\omega \in \Omega$ of a given family $\{Y(\theta)\}_{\theta \in \Theta}$ of functions $Y(\theta) : \Omega \rightarrow S$ parametrized by θ by seeing them as sections of functions depending on two parameters. So if, for instance, $D_0(\theta)$ is the function introduced in Theorem 19.1, we will write

$$D_0(\theta, \omega) := D_0(\theta)(\omega)$$

and so on.

The following result basically follows from Theorem 19.1 via Theorem 11.1. We state it in a language that will be convenient for our forthcoming proofs.

Corollary 19.2 (The Averaged-frequency Quenched Invariance Principle for Martingales). *Assume that $D_0(\theta) \in L^2_{\mathbb{P}}(\mathcal{F}_0) \ominus L^2_{\mathbb{P}}(\mathcal{F}_{-1})$ is given for every $\theta \in [0, 2\pi)$, and that the function $(\theta, \omega) \mapsto D_0(\theta, \omega)$ is $\mathcal{B} \otimes \mathcal{F}$ -measurable (see Remark 19.2). Then, with the notation in Theorem 19.1, and assuming that \mathcal{F} is countably generated, there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}\Omega_0 = 1$ such that for all $\omega_0 \in \Omega_0$, the distribution of $(\theta, \omega) \mapsto V_n(\theta, \omega)$ under $\lambda \times \mathbb{P}_{\omega_0}$ converges to that of $(\theta, \omega') \mapsto B(\theta, \omega')$ under $\lambda \times \mathbb{P}'$.*

Proof: First, the $\mathcal{B} \otimes \mathcal{F}$ -(resp. $\mathcal{B} \otimes \mathcal{F}'$)-measurability of $(\theta, \omega) \mapsto V_n(\theta, \omega)$ (resp. $(\theta, \omega') \mapsto B(\theta, \omega')$) follows at once from Remark 7.1 (page 49).

We claim that given any continuous and bounded function $f : D[[0, \infty), \mathbb{C}] \rightarrow \mathbb{R}$

$$\lim_n E[f \circ V_n | \mathcal{B} \otimes \mathcal{F}_0](\theta, \omega) = Ef(B(\theta)) \quad (5.21)$$

$\lambda \times \mathbb{P}$ -a.s., where the expectation at the left-hand side (resp. right-hand side) denotes integration with respect to \mathbb{P} (resp. \mathbb{P}').

Before proceeding to the proof of (5.21), let us explain why this implies the desired conclusion:

1. First, note that (5.21) can be considered an equality of $\mathcal{B} \otimes \mathcal{F}_0$ measurable functions, the \mathcal{B} -measurable function at the right being considered as constant in Ω for fixed θ .
2. It follows by an application of Theorem 11.1 that, for any given $\mathcal{B}_0 \subset \mathcal{B}$

$$\lim_n E[f \circ V_n | \mathcal{B}_0 \otimes \mathcal{F}_0] = E[Ef(B(\theta)) | \mathcal{B}_0 \otimes \mathcal{F}_0] \quad (5.22)$$

$\lambda \times \mathbb{P}$ -a.s.

3. If $\mathcal{B}_0 = \{\emptyset, [0, 2\pi)\}$ is the trivial sigma algebra then (see Example 4 in page 64) if we define $\lambda_\theta := \lambda$ for all $\theta \in [0, 2\pi)$, $\{\lambda_\theta\}_{\theta \in [0, 2\pi)}$ is a decomposition of $E[\cdot | \mathcal{B}_0]$ and it follows, from Proposition 14.1, that (5.22) is nothing but the statement of convergence $V_n \Rightarrow B$ under $\lambda \times \mathbb{P}_\omega$ for \mathbb{P} -a.e ω : this is the desired conclusion.

Proof of (5.21). To prove (5.21) we proceed as follows: first, the set

$$\{(\theta, \omega) : \lim_n (E[f \circ V_n | \mathcal{B} \otimes \mathcal{F}_0](\theta, \omega) - Ef(B(\theta))) = 0\}$$

is $\mathcal{B} \otimes \mathcal{F}$ measurable, and to see that it has product measure one it suffices to see that for λ -a.e fixed θ

$$\mathbb{P}[\lim_n (E[f \circ V_n | \mathcal{B} \otimes \mathcal{F}_0](\theta, \cdot) - Ef(B(\theta))) = 0] = 1. \quad (5.23)$$

Let I be the set

$$I := \{\theta \in [0, 2\pi) : e^{2i\theta} \notin \text{Spec}_p(T)\},$$

which satisfies $\lambda(I) = 1$ according to Proposition 1.4 (\mathcal{F} is countably generated). We claim that (5.23) holds for every $\theta \in I$.

To see why this claim is true, note that by Proposition 14.1 and Example 4 again, if δ_θ denotes the Dirac measure at θ , then

$$\{\delta_\theta \times \mathbb{P}_\omega\}_{(\theta, \omega) \in [0, 2\pi) \times \Omega}$$

is a decomposition of $E[\cdot | \mathcal{B} \times \mathcal{F}_0]$, and Theorem 19.1 gives that for every $\theta \in I$ there exists Ω_θ with $\mathbb{P}\Omega_\theta = 1$ such that for every $\omega \in \Omega_\theta$

$$\lim_n E[f \circ V_n | \mathcal{B} \otimes \mathcal{F}_0](\theta, \omega) = \lim_n \int_{[0, 2\pi) \times \Omega} f \circ V_n(\alpha, z) d(\delta_\theta \times \mathbb{P}_\omega)(\alpha, z) =$$

$$\lim_n \int_\Omega f(V_n(\theta, z)) d\mathbb{P}_\omega(z) = \lim_n E[f(V_n(\theta)) | \mathcal{F}_0](\omega) = Ef(B(\theta))$$

as desired. □

Chapter 6

Proofs of Theorems 15.1, 17.1 and 17.2

The exposition is divided as follows: Section 20 presents the martingale approximation results leading to the proof of the theorems stated in the title of this chapter. This section is divided into two parts: “Approximation Lemmas” (Section 20.1), giving a presentation of the abstract martingale approximation results that will be used to construct the proofs of the corresponding theorems, and “The Approximating Martingales” (Section 20.2), in which we present the actual martingales to be used along the rest of the chapter.

Section 21 presents the proof of Theorem 15.1 which, in analogy with the forthcoming proofs, consists of verifying the hypothesis of the corresponding lemma from Section 20.1 via the martingales introduced in Section 20.2. The key step is a further, “concrete” approximation lemma (Lemma 9), whose proof at some point makes use of a technique analogous to that used to prove Theorem 3.2. With such lemma and the previous results at hand, the proof of the aforementioned theorem is reduced to a few, almost obvious, lines.

Section 22 is devoted to the proofs of theorems 17.1 and 17.2. The reason to present these proofs in the same section lies in the fact that, as the reader will see, the corresponding arguments can be considered “branches” of the same decomposition of the difference between the process and the approximating martingales (Lemma 10), and in particular to stress the “smoothing” role of Hunt and Young’s inequality (Theorem 2.3) in the proofs involving “averaged” (as opposed to “fixed”) frequencies.

The chapter finishes with a note (see page 115) pointing out that the use of Theorem 10.1 along these proofs is not essential.

20 Martingale Approximations

In this section we will give a series of approximation lemmas whose verification will imply the results stated as theorems 15.1, 17.1 and 17.2. For the sake of clarity, we will limit our discussion in this section to state and prove the aforementioned lemmas and in particular to explain *why* these imply the corresponding results stated in Chapter 4. We will also present, without further analysis, the martingales used along the proofs. The actual verification of the hypotheses in these lemmas under the hypotheses of the corresponding theorems via the given martingales is deferred to later sections.

20.1 Approximation Lemmas

Our first approximation lemma is the following.

Lemma 6 (Approximation Lemma for Theorem 15.1). *Under the hypotheses and notation in Theorem 15.1, assume that there exists $I' \subset [0, 2\pi)$ with $\lambda(I') = 1$ satisfying the following: for every $\theta \in I'$, there exists $D_0(\theta) \in L^2(\mathcal{F}_0) \ominus L^2(\mathcal{F}_{-1})$ with the property that, if we denote $M_n(\theta) := \sum_{k=0}^{n-1} T^k D_0(\theta) e^{ik\theta}$ ($n \in \mathbb{N}^*$),*

$$\lim_n \frac{1}{n} E_0 |S_n(\theta) - E_0 S_n(\theta) - M_n(\theta)|^2 = 0 \quad (6.1)$$

\mathbb{P} -a.s. and in $L^1_{\mathbb{P}}$. Then the conclusion of Theorem 15.1 holds with $I = I' \setminus \{\theta : e^{2i\theta} \in \text{Spec}_p(T)\}$ and

$$\sigma^2(\theta) = E |D_0(\theta)|^2. \quad (6.2)$$

Before proving this lemma let us point out the following interesting fact: assume that, for $\theta \in [0, 2\pi)$, $D_0(\theta)$ and $D'_0(\theta)$ are given as in Lemma 6, and let $(M_n(\theta))_{n \in \mathbb{N}^*}$ and $(M'_n(\theta))_{n \in \mathbb{N}^*}$ be the corresponding $(\mathcal{F}_{n-1})_{n \in \mathbb{N}^*}$ -adapted martingales. Then, according to Corollary 4.2 and the footnote in Remark 19.1

$$\begin{aligned} E |D_0(\theta) - D'_0(\theta)|^2 &= \lim_n E_0 \frac{1}{n} |M_n(\theta) - M'_n(\theta)|^2 \leq \\ 2 \limsup_n \frac{1}{n} (E_0 |S_n(\theta) - E_0 S_n(\theta) - M_n(\theta)|^2 &+ E_0 |S_n(\theta) - E_0 S_n(\theta) - M'_n(\theta)|^2). \end{aligned} \quad (6.3)$$

In particular, we have the following uniqueness result.

Proposition 20.1 (Uniqueness of $D_0(\theta)$). *In the context of Lemma 6, and given $\theta \in [0, 2\pi)$ (not necessarily in I'), there exists at most one function $D_0(\theta) \in L^2_{\mathbb{P}}(\mathcal{F}_0) \ominus L^2_{\mathbb{P}}(\mathcal{F}_{-1})$ satisfying (6.1).*

Proof: Combine (6.1) with (6.3). □

We proceed now to the proof of Lemma 6.

Proof of Lemma 6: First, note that $\lambda(I) = 1$ by Proposition 1.4.

Let now $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$ be a decomposition of E_0 (Definition 11.2). According to (6.1), there exists, for $\theta \in I$, $\Omega_\theta \subset \Omega$ with $\mathbb{P}\Omega_\theta = 1$ such that, for all $\omega \in \Omega_\theta$

$$\lim_n \frac{1}{n} \|S_n(\theta) - E_0 S_n(\theta) - M_n(\theta)\|_{\mathbb{P}_\omega, 2}^2 = 0. \quad (6.4)$$

and the quenched convergence stated in Theorem 15.1 follows at once from Theorem 19.1 (taking $t = 1$), Proposition 13.1, and Corollary 10.2 (replacing $X_{r,n} := M_n(\theta)/\sqrt{n}$ and $X_n := (S_n(\theta) - E_0 S_n(\theta))/\sqrt{n}$).

Now, by orthogonality under E_0 (see the footnote in Remark 19.1) and Corollary 4.2,

$$E[|D_0(\theta)|^2] = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} E_0 T^k |D_0(\theta)|^2 = \lim_n \frac{1}{n} E_0 |M_n(\theta)|^2$$

in the \mathbb{P} -a.s and $L^1_{\mathbb{P}}$ senses, which implies by (6.4) and the Minkowski inequality that

$$\lim_n \frac{1}{n} E_0 |S_n(\theta) - E_0 S_n(\theta)|^2 = E[|D_0(\theta)|^2] \quad (6.5)$$

\mathbb{P} -a.s. and in $L^1_{\mathbb{P}}$: this is the statement 1. in Theorem 15.1.

Finally, to see that $\theta \mapsto \sigma^2(\theta)$ necessarily defines a version of the spectral density of $(X_k - E_{-\infty} X_k)_{k \in \mathbb{Z}}$ we proceed as follows: integrating (6.5) and using the $L^1_{\mathbb{P}}$ convergence we get that, for $\theta \in I$

$$\begin{aligned} E[|D_0(\theta)|^2] &= \lim_n \frac{1}{n} E |S_n(\theta) - E_0 S_n(\theta)|^2 = \\ &= \lim_n \frac{1}{n} E |S_n(\theta) - E_{-\infty} S_n(\theta) - E_0(S_n(\theta) - E_{-\infty} S_n(\theta))|^2 = \\ &= \lim_n \frac{1}{n} E |(S_n(\theta) - E_{-\infty} S_n(\theta))|^2, \end{aligned} \quad (6.6)$$

where for the last equality we used the fact that

$$\lim_n \frac{1}{n} E |E_0(S_n(\theta) - E_{-\infty} S_n(\theta))|^2 = 0$$

(see the proof of Corollary 15.3). The conclusion follows from (6.6), Theorem 5.4 and the fact that $\lambda(I) = 1$. \square

Our next two approximation lemmas make use of an additional parameter, “ r ”, whose presence will allow us in particular to carry on the proofs of Theorems 17.1 and 17.2 without restricting ourselves explicitly to the set I in Lemma 6.¹

Lemma 7 (Approximation Lemma for Theorem 17.2). *With the notation and conventions introduced on page 89, and with the additional notation (4.17) and (4.18), let $\theta \in [0, 2\pi)$*

¹Nevertheless, we will work under this restriction when carrying on the actual proofs.

be such that $e^{2i\theta} \notin \text{Spec}_p(T)$. Assume given, for every $r \in \mathbb{N}$, a function $D_{r,0}(\theta) \in L^2_{\mathbb{P}}(\mathcal{F}_0) \ominus L^2_{\mathbb{P}}(\mathcal{F}_{-1})$, and given $n \in \mathbb{N}^*$ denote by $M_{r,n}(\theta)$ the function

$$M_{r,n}(\theta) := \sum_{k=0}^{n-1} T^k D_{r,0}(\theta) e^{ik\theta}.$$

Then the hypothesis

$$\lim_r \limsup_n E_0 \left[\frac{1}{n} \max_{1 \leq k \leq n} |S_k(\theta) - E_0 S_k(\theta) - M_{r,n}(\theta)|^2 \right] = 0, \quad \mathbb{P}\text{-a.s.}, \quad (6.7)$$

implies the existence of

$$\sigma^2(\theta) := \lim_r E[|D_{r,0}^2(\theta)|], \quad (6.8)$$

and if we denote

$$B(\theta)(\omega') := (\sigma^2(\theta)/2)^{1/2} (B_1(\omega') + iB_2(\omega')), \quad (6.9)$$

then $W_n(\theta)$ converges in the quenched sense (with respect to \mathcal{F}_0) to $B(\theta)$ as $n \rightarrow \infty$.

Before proceeding to the proof of Lemma 7, let us point out the following.

Remark 20.1 (Consistency of the Notation (6.8)). Notice that, in the context of Lemma 6, if for $\theta \in I$ the hypotheses of Lemma 7 are verified, then necessarily

$$\lim_r E[|D_{r,0}(\theta)|^2] = E|D_0(\theta)|^2,$$

where $D_0(\theta)$ is chosen according to Lemma 6.

To see this just note that, for such θ , the conclusion of Lemma 6 follows from Lemma 7 by evaluating (6.9) at $t = 1$, and compare the corresponding random variables thus obtained.²

Proof of Lemma 7: Start by recalling the notation and criteria introduced in Section 7.2, specially in the numeral 4., and define $V_{r,n}$ as in (5.6) with $D_{r,0}$ in place of D_0 for every $(r, n) \in \mathbb{N} \times \mathbb{N}^*$.

For $m \geq 1$, the Skorohod metric d_m on $D[[0, m], \mathbb{C}]$, is dominated by the uniform (product) metric. Thus for every $(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$

$$d_m(r_m W_n(\theta, \omega), r_m V_{r,n}(\theta, \omega)) \leq \frac{\sqrt{m}}{\sqrt{n'}} \max_{1 \leq k \leq n'} |S_k(\theta, \omega) - E_0 S_k(\theta)(\omega) - M_{r,n}(\theta, \omega)|.$$

where $n' = mn$. It follows from (6.7) that there exists $\Omega_{0,1} \subset \Omega$ with $\mathbb{P}\Omega_{0,1} = 1$ such that if $\omega \in \Omega_{0,1}$

$$\lim_r \limsup_n \|d_m(r_m W_n(\theta), r_m V_{r,n}(\theta))\|_{\mathbb{P}_\omega, 2} = 0. \quad (6.10)$$

²If X, Y are nonzero random variables, $X = Y$ in distribution, and $a, b \geq 0$ are constants with $aX = bY$ in distribution, then $a = b$: for every $M > 0$, $0 = bE[|Y|I_{[|bY| \leq M]}] - aE[|X|I_{[|aX| \leq M]}] = (b - a)E[|X|I_{[|aX| \leq M]}]$.

Now, according to Theorem 19.1 and Proposition 13.1, there exists $\Omega_{0,2} \subset \Omega$ with $\mathbb{P}\Omega_{0,2} = 1$ with the following property: for every $\omega \in \Omega_{0,2}$,

$$V_{r,n}(\theta) \Rightarrow_n B_r(\theta)$$

under \mathbb{P}_ω where $B_r(\theta)$ is the random element with domain in $(\Omega', \mathcal{F}', \mathbb{P}')$ defined by

$$B_r(\theta)(\omega') := (E[|D_{r,0}(\theta)|^2]/2)^{1/2}(B_1(\omega') + iB_2(\omega')). \quad (6.11)$$

Since for every fixed $m \geq 0$, B_r is \mathbb{P}' -a.e continuous at m , the observations in Section 7.2 (numeral 4. again) imply that for every $m \in \mathbb{N}$ and every $\omega \in \Omega_{0,2}$

$$r_m V_{r,n}(\theta) \Rightarrow_n r_m B_r(\theta) \quad (6.12)$$

under \mathbb{P}_ω as $n \rightarrow \infty$.

Let $\Omega_0 := \Omega_{0,1} \cap \Omega_{0,2}$. According to Theorem 19.1 and Corollary 10.2, (6.10) together with (6.12) imply the following: given $\omega \in \Omega_0$ and $m > 0$, there exists a random element $\hat{B}^m(\theta)$ of $D[[0, m], \mathbb{C}]$ such that

$$r_m W_n(\theta) \Rightarrow_n \hat{B}^m \quad (6.13)$$

under \mathbb{P}_ω , and $r_m B_r(\theta) \Rightarrow_r \hat{B}^m(\theta)$ under \mathbb{P}' .

We claim that, actually, there exists

$$\sigma^2(\theta) := \lim_r E[|D_{r,0}^2(\theta)|]$$

from where it follows easily that, if $B(\theta)$ is given by (6.9), the distribution of $\hat{B}^m(\theta)$ is the same as that of $r_m B(\theta)$, and the conclusion will follow at once from 4. in Section 7.2, (6.13) and Proposition 13.1, because $\mathbb{P}\Omega_0 = 1$.

Proof of the existence of (6.8). To prove the existence of the limit (6.8) notice first that, by Theorem 7.1 there exists, for every $m > 0$, a number $0 < t < m$ such that $r_m B_r(\theta)(t) \Rightarrow_r \hat{B}^m(\theta)(t)$. For any of such t we get the existence of a random variable $N(\theta, t)$ such that, if N_1 and N_2 are i.i.d standard normal variables

$$\left(\frac{tE|D_{0,r}(\theta)|^2}{2} \right)^{1/2} (N_1 + iN_2) \Rightarrow_r N(\theta, t)$$

and the existence of the limit in (6.8) follows at once from Proposition 8.3 in page 53.

Finally, note that $\sigma(\theta)$ is indeed given by (4.2) in accordance to Remark 20.1 and the statement of Lemma 6. \square

Before proceeding to the next approximation lemma let us anticipate the fact that, under (4.22), the hypotheses of Lemma 7 will be verified for every $\theta \in [0, 2\pi)$ provided that $e^{2i\theta} \notin \text{Spec}_p(T)$. On proving this, we will encounter some “intermediate” approximations that will lead us to verify the hypotheses of Lemma 8 below assuming *only* the hypotheses of Theorem 17.1.

Our next approximation lemma is the “two-parameters” version of the previous one.

Lemma 8 (Approximation Lemma for the Proof of Theorem 17.1). *Under the setting in page 89, denote by $\mathcal{D}_{\infty, \mathbb{C}}$ the Borel sigma-algebra of $D[[0, \infty), \mathbb{C}]$. Assume that for every $(r, \theta) \in \mathbb{N} \times [0, 2\pi)$, $D_{r,0}(\theta)$ is given as in the statement of Lemma 7 and that the function $(\theta, \omega) \mapsto D_{r,0}(\theta)(\omega)$ is $\mathcal{B} \otimes \mathcal{F}_\infty / \mathcal{D}_{\infty, \mathbb{C}}$ measurable, and denote by E_0 the version of $E[\cdot | \mathcal{F}_0]$ given by integration with respect to $\{\mathbb{P}_\omega\}_{\omega \in \Omega}$ (Definition 11.2): $E_0 X(\omega) = E^\omega X$ for every $X \in L^1_{\mathbb{P}}$. Then the hypotheses*

1. *There exists $I' \subset [0, 2\pi)$ with $\lambda(I') = 1$ such that, for every $\theta \in I'$*

$$\sigma^2(\theta) := \lim_r E|D_{r,0}(\theta)|^2$$

is well defined.

2. *The equality*

$$\lim_r \limsup_n \int_0^{2\pi} E^\omega \left[\frac{1}{n} \max_{1 \leq k \leq n} |S_k(\theta, \cdot) - E_0[S_k(\theta, \cdot)] - M_{r,k}(\theta, \cdot)|^2 \right] d\lambda(\theta) = 0 \quad (6.14)$$

(see also Remark 19.2) holds for \mathbb{P} -a.e ω .

imply (together) the conclusion of Theorem 17.1.

Proof: First, the assumption that $E_0[Z](\omega) = E^\omega Z$ where E^ω denotes integration with respect to \mathbb{P}_ω guarantees the $\mathcal{B} \otimes \mathcal{F}_0$ -measurability of the integrand (see Step 2. in the proof of Proposition 14.1). In particular, by [11], Theorem 18.1-(ii), the given integral makes sense for every $\omega \in \Omega$.

Now note that, since for every $(k, r, \theta) \in \mathbb{Z} \times \mathbb{N} \times [0, 2\pi)$ the random variables $T^k D_{r,0}(\theta, \cdot)$ and $T^k X_0$ are \mathcal{F}_∞ -measurable, we can assume that $\mathcal{F} = \mathcal{F}_\infty$. Since \mathcal{F}_∞ is countably generated (\mathcal{F}_0 is), Corollary 19.2 (page 97) guarantees that there exist $\Omega_{0,1}$ with $\mathbb{P}\Omega_{0,1} = 1$ such that for every $\omega \in \Omega_{0,1}$, $(\theta, \omega) \mapsto V_{r,n}(\theta, \omega)$ converges to (6.11) under $\lambda \times \mathbb{P}_\omega$.

The same arguments as in the proof of Lemma 8 guarantee that there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}\Omega_0 = 1$ such that, for every fixed $r \in \mathbb{N}$, the sequence of $\mathcal{B} \otimes \mathcal{F} / \mathcal{D}_{\infty, \mathbb{C}}$ -measurable functions $(\theta, \omega) \mapsto V_{r,n}(\theta, \omega)$ satisfy $V_{r,n} \Rightarrow_n B_r$ under $\lambda \times \mathbb{P}_\omega$ (where $(\theta, \omega') \mapsto B_r(\theta, \omega')$ is given by (6.11)), that there exists a random function $\hat{B} \in D[[0, \infty), \mathbb{C}]$ (defined on some unspecified probability space) such that $B_r \Rightarrow_r \hat{B}$ (under $\lambda \times \mathbb{P}'$) and that, for every $\omega \in \Omega_0$, $W_n \Rightarrow_n \hat{B}$ under $\lambda \times \mathbb{P}_\omega$.

To prove that we can take $\hat{B} = B$, where B is as in the statement of Theorem 17.1, note that the λ -a.e well definition of (6.8) guarantees that B_r converges to

$$B(\theta, \omega') = (\sigma^2(\theta)/2)^{1/2} (B_1(\omega') + B_2(\omega')),$$

$\lambda \times \mathbb{P}'$ -a.s. and that, according to Remark 20.1 and Lemma 6, $\theta \mapsto \sigma^2(\theta)$ is certainly a version of the spectral density of $(X_k - E_{-\infty} X_k)_{k \in \mathbb{Z}}$. \square

20.2 The Approximating Martingales

We finish this section introducing the martingales used along the proofs of the results established in this chapter. We will defer any discussion about the martingales themselves to later sections.

For every $(r, n, \theta) \in \mathbb{N} \times \mathbb{N}^* \times [0, 2\pi)$, denote

$$\begin{aligned} D_{r,0}(\theta) &:= \sum_{k=0}^r \mathcal{P}_0 X_k e^{ik\theta}, & M_{r,n}(\theta) &:= \sum_{k=0}^{n-1} T^k D_{r,0}(\theta) e^{ik\theta} \\ D_0(\theta) &:= \lim_r D_{r,0}(\theta), & M_n(\theta) &:= \sum_{k=0}^{n-1} T^k D_0(\theta) e^{ik\theta}. \end{aligned} \quad (6.15)$$

When necessary, we will indicate the dependence on X_0 , T , and \mathcal{P}_0 by denoting

$$D_{r,0}(\theta) = D_{r,0}(X_0, T, \mathcal{P}_0, \theta), \quad (6.16)$$

and so on.

where $D_0(\theta)$ is defined as a limit in the $L_{\mathbb{P}}^2$ sense, provided that such limit exists.

In consonance with the notation introduced in Definition 2.6, we treat the case $\theta = 0$ denoting, for every $r \in \mathbb{N}$,

$$D_{r,0} := D_{r,0}(0), \quad D_0 := D_0(0) \quad \text{and} \quad M_{r,n} := M_{r,n}(0), \quad M_n := M_n(0). \quad (6.17)$$

21 Proof of Theorem 15.1

The following lemma will be of fundamental importance to prove the validity of the hypotheses of Lemma 6.

Lemma 9 (Almost Surely Approximation Lemma). *In the context of Theorem 15.1, and with the notation (6.15) and (6.17), fix $\theta \in [0, 2\pi)$ and assume that $D_{r,0}(\theta)$ converges \mathbb{P} -a.s. as $r \rightarrow \infty$ and that $\sup_{r \in \mathbb{N}} |D_{r,0}(\theta)| \in L_{\mathbb{P}}^2$. Then $D_0(\theta)$ is well defined and*

$$\lim_n \frac{1}{n} E_0 |S_n(\theta) - E_0 S_n(\theta) - M_n(\theta)|^2 = 0. \quad (6.18)$$

\mathbb{P} -a.s. and in $L_{\mathbb{P}}^1$.

Proof. We will proceed in two steps.

Step 1. Assume $\theta = 0$. We will prove that, if $\sup_{r \in \mathbb{N}} |D_{r,0}| \in L_{\mathbb{P}}^2$ and $D_{r,0}$ converges \mathbb{P} -a.s. as $r \rightarrow \infty$, then

$$\lim_n \frac{1}{n} E_0 |S_n - E_0 S_n - M_n|^2 = 0,$$

\mathbb{P} -a.s.

Let $D_0 := \lim_r D_{r,0}$ (in the \mathbb{P} -a.s. sense). To see that $\lim_r D_{r,0} = D_0$ in $L_{\mathbb{P}}^2$ note that $|D_{r,0} - D_0| \leq 2 \sup_{r \in \mathbb{N}} |D_{r,0}|$ and therefore, by the dominated convergence theorem and the hypotheses on $(D_{r,0})_{r \in \mathbb{N}}$,

$$\lim_r E |D_0 - D_{r,0}|^2 = 0.$$

as desired. Note also that, by a similar argument

$$\lim_N E [\sup_{j \geq N} |D_0 - D_{j,0}|^2] = 0. \quad (6.19)$$

Notice now that $D_0 \in L^2_{\mathbb{P}}(\mathcal{F}_0) \ominus L^2_{\mathbb{P}}(\mathcal{F}_{-1})$, because this is a closed subspace of $L^2_{\mathbb{P}}$. To prove (6.18) let us start in the following way: given $n \in \mathbb{N}^*$, we have

$$S_n - E_0 S_n - M_n = \sum_{k=0}^{n-1} (E_k S_n - E_{k-1} S_n - T^k D_0) = \sum_{k=0}^{n-1} \mathcal{P}_k T^k (S_{n-k} - D_0).$$

The term at the right-hand side is a decomposition of the term at the left-hand side as a sum of orthogonal functions with respect to E_0 (see the footnote on Remark 19.1), and therefore

$$\frac{1}{n} E_0 [|S_n - E_0 S_n - M_n|^2] = \frac{1}{n} \sum_{k=1}^n E_0 [|\mathcal{P}_k T^k (S_{n-k} - D_0)|^2] = \frac{1}{n} \sum_{k=1}^n E_0 T^k |D_0 - D_{n-k,0}|^2.$$

Now fix $N \in \mathbb{N}^*$. For every $n \geq N$ we can decompose

$$\frac{1}{n} \sum_{k=1}^n E_0 T^k |D_0 - D_{n-k,0}|^2 = \frac{1}{n} \sum_{k=1}^{n-N} E_0 T^k |D_0 - D_{n-k,0}|^2 + \sum_{k=n-N+1}^n E_0 T^k |D_0 - D_{n-k,0}|^2 \leq$$

$$\frac{1}{n} \sum_{k=1}^{n-N} E_0 T^k \sup_{j \geq N} |D_0 - D_{k,0}|^2 + \frac{2}{n} \sum_{k=n-N+1}^n E_0 T^k \sup_{j \geq 0} |D_{j,0}|^2.$$

Note that, since the last summand contains (only) the last N elements of the $n+1$ -th ergodic average in the statement of Theorem 4.1 corresponding to the random variable $\sup_{j \geq 0} |D_{j,0}|^2$ we have, according to such result combined with the estimates above, that

$$\limsup_{n \in \mathbb{N}} \frac{1}{n} E_0 |S_n - E_0 S_n - M_n|^2 \leq E_0 P_0 [\sup_{j \geq N} |D_0 - D_{k,0}|^2],$$

both in the \mathbb{P} -a.s. and $L^1_{\mathbb{P}}$ senses, where P_0 is the orthogonal projection over the subspace of T -invariant functions (see Remark 3.3). The conclusion follows via (6.19) and the continuity in $L^1_{\mathbb{P}}$ of $E_0 P_0$ by letting $N \rightarrow \infty$.

Step 2: general case. The general case follows from the previous one via the following argument: assume that the hypotheses in Lemma 9 hold for a given $\theta \in [0, 2\pi)$, and denote by \tilde{E} the integration with respect to $\lambda \times \mathbb{P}$. Let \tilde{X}_0 be the extension to the product space specified by Definition 3.3, let \tilde{T}_θ be the extension map in (1.30) and, for every $k \in \mathbb{Z}$, let $\tilde{\mathcal{F}}_k = \mathcal{B} \otimes \mathcal{F}_k$ and $\tilde{E}_k := E[\cdot | \tilde{\mathcal{F}}_k]$, so that for $Y \in L^1_{\mathbb{P}}$,

$$\tilde{E}_k \tilde{Y}(u, \omega) := E[\tilde{Y} | \tilde{\mathcal{F}}_k](u, \omega) = e^{iu} E_k Y(\omega) = \widetilde{E_k Y}(u, \omega).$$

Note also that, since $|\tilde{Y}(u, \omega)|^2 = |Y(\omega)|^2$, then $\tilde{E}_k |\tilde{Y}|^2(u, \omega) = E_k |Y|^2(\omega)$.

It is not hard to see that $(\tilde{\mathcal{F}}_k)_{k \in \mathbb{Z}}$ is a \tilde{T}_θ -filtration (Definition 4.2), that $\tilde{X}_0 \in L^2_{\lambda \times \mathbb{P}}(\tilde{\mathcal{F}}_0)$, and that if we follow the definitions in (6.15) and (6.17) (see also (6.16)) with \tilde{X}_0 , \tilde{T}_θ , and \tilde{E}_k in place of X_0 , T and E_k then we get that

$$D_{r,0}(\tilde{X}_0, \tilde{T}_\theta, \tilde{\mathcal{P}}_0, 0) = \tilde{D}_{r,0}(X_0, T, \mathcal{P}_0, \theta),$$

and similarly for $M_{r,n}(\tilde{X}_0, \tilde{T}_\theta, \tilde{P}_0, 0)$ and $S_n(\tilde{X}_0, \tilde{T}_\theta, 0)$.

In particular, $\tilde{E}[\sup_{r \in \mathbb{N}} |\tilde{D}_{r,0}(\theta)|^2] = E[\sup_{r \in \mathbb{N}} |D_{r,0}(\theta)|^2] < \infty$ and

$$\tilde{D}_{r,0}(\theta)(u, \omega) = e^{iu} D_{r,0}(\theta)(\omega)$$

converges $\lambda \times \mathbb{P}$ -a.s.

Finally, by the case already studied ($\theta = 0$), we have that for $\lambda \times \mathbb{P}$ -a.e (u, ω) ,

$$0 = \lim_n \frac{1}{n} \tilde{E}_0 |\tilde{S}_n(\theta) - \tilde{E}_0 \tilde{S}_n(\theta) - \tilde{M}_n(\theta)|^2(u, \omega) = \lim_n \frac{1}{n} E_0 |S_n(\theta) - E_0 S_n(\theta) - M_n(\theta)|^2(\omega), \quad (6.20)$$

which implies the desired conclusion by fixing u in such a way that the first equality holds \mathbb{P} -a.s. \square

Lemma 9 completes the set of tools needed to reach the proof of Theorem 15.1.

Proof of Theorem 15.1: Note that, according to (1.47),

$$T^k \mathcal{P}_{-k} X_0 = \mathcal{P}_0 X_k$$

for every $k \in \mathbb{Z}$ and therefore, since T is measure preserving

$$\|X_0\|_{\mathbb{P},2}^2 = \sum_{k \geq 0} \|\mathcal{P}_{-k} X_0\|_{\mathbb{P},2}^2 = \sum_{k \geq 0} \|\mathcal{P}_0 X_k\|_{\mathbb{P},2}^2. \quad (6.21)$$

An application of Proposition 2.3 (page 20) combined with lemmas 6 and 9 gives the conclusion in Theorem 15.1. \square

22 Proof of Theorems 17.1 and 17.2

We move on now to the construction of proofs for theorems 17.1 and 17.2.

Following the explanations in Section 20.1, our goal is to prove the approximations (6.7) and (6.14) in lemmas 7 and 8. Our first step towards this goal is to prove the following decomposition.

Lemma 10. *In the setting on page 89, for all $(n, r, \theta) \in \mathbb{N} \times \mathbb{N}^* \times [0, 2\pi)$, $X_0 \in L_{\mathbb{P}}^2(\mathcal{F}_0)$, and with the notation (6.15), the following equality holds :*

$$\begin{aligned} S_n(\theta) - E_0 S_n(\theta) - M_{r,n}(\theta) &= -e^{i(n-1)\theta} \left(\sum_{k=1}^r (T^{n-1} E_0 X_k - E_0 T^{n-1} E_0 X_k) e^{ik\theta} \right) \\ &+ e^{ir\theta} \sum_{k=2}^{n-1} (T^k E_{-1} X_r - E_0 T^k E_{-1} X_r) e^{ik\theta} \\ &- D_{r,0}(\theta). \end{aligned} \quad (6.22)$$

Proof: Fix $(n, r, \theta) \in \mathbb{N} \times \mathbb{N}^* \times [0, 2\pi)$. We depart from the following decomposition of X_0 (the array is intended to make visible the rearrangements):

$$\begin{aligned}
X_0 = E_0 X_0 &= (E_0 - E_{-1})X_0 + E_{-1}X_0 \\
&+ (E_0 - E_{-1})X_1 e^{i\theta} - (E_0 - E_{-1})X_1 e^{i\theta} + \\
&+ (E_0 - E_{-1})X_2 e^{i2\theta} - (E_0 - E_{-1})X_2 e^{i2\theta} + \\
&\vdots \\
&+ (E_0 - E_{-1})X_r e^{ir\theta} - (E_0 - E_{-1})X_r e^{ir\theta} \\
&= \sum_{k=0}^r (\mathcal{P}_0 X_k) e^{ik\theta} - \sum_{k=1}^r (E_0 X_k e^{ik\theta} - E_{-1} X_{k-1} e^{i(k-1)\theta}) \\
&\quad + E_{-1} X_r e^{ir\theta}.
\end{aligned} \tag{6.23}$$

Now, using the equality

$$\sum_{j=0}^{n-1} e^{ij\theta} T^j \sum_{k=1}^r (E_0 X_k e^{ik\theta} - E_{-1} X_{k-1} e^{i(k-1)\theta}) = e^{i(n-1)\theta} T^{n-1} \sum_{k=1}^r E_0 X_k e^{ik\theta} - \sum_{k=0}^{r-1} E_{-1} X_k e^{ik\theta}$$

we get, from (6.23), that

$$\begin{aligned}
S_n(\theta) &= M_{r,n}(\theta) - (e^{i(n-1)\theta} T^{n-1} \sum_{k=1}^r E_0 X_k e^{ik\theta} - \sum_{k=0}^{r-1} E_{-1} X_k e^{ik\theta}) \\
&+ \sum_{j=0}^{n-1} e^{ij\theta} T^j E_{-1} X_r e^{ir\theta}
\end{aligned} \tag{6.24}$$

and that

$$\begin{aligned}
E_0 S_n(\theta) &= D_{r,0}(\theta) - (E_0 e^{i(n-1)\theta} T^{n-1} \sum_{k=1}^r E_0 X_k e^{ik\theta} - \sum_{k=0}^{r-1} E_{-1} X_k e^{ik\theta}) \\
&+ \sum_{j=0}^{n-1} e^{ij\theta} E_0 T^j E_{-1} X_r e^{ir\theta}
\end{aligned} \tag{6.25}$$

(6.22) follows from (6.24) and (6.25) (see also Proposition 4.1). \square

The next step towards (6.14) lies in the use of appropriate upper bounds for the terms at the right-hand side in (6.22): for a given $(n, r, \theta) \in \mathbb{N} \times \mathbb{N}^* \times [0, 2\pi)$ let us denote by $A_{r,n} : [0, 2\pi) \times \Omega \rightarrow \mathbb{C}$ and $B_{r,n} : [0, 2\pi) \times \Omega \rightarrow \mathbb{C}$ the $\mathcal{B} \otimes \mathcal{F}_\infty$ -measurable functions

$$A_{r,n}(\theta, \omega) := \sum_{k=1}^r (T^{n-1} E_0 X_k(\omega) - E_0 T^{n-1} E_0 X_k(\omega)) e^{ik\theta}, \tag{6.26}$$

$$B_{r,n}(\theta, \omega) := \sum_{k=0}^{n-1} (T^k E_{-1} X_r(\omega) - E_0 T^k E_{-1} X_r(\omega)) e^{ik\theta}. \tag{6.27}$$

Then we have the following lemma.

Lemma 11. *In the context of Lemma 10, and with the notation (6.26) and (6.27), there exists a constant $C > 0$ such that, if E_0 is given by the regular version $E_0 X(\omega) = E^\omega X$ ($X \in L^1_{\mathbb{P}}$) then*

1. For all $(n, r, \theta) \in \mathbb{N} \times \mathbb{N}^* \times [0, 2\pi)$, $\alpha \in \mathbb{R}$, and $\omega \in \Omega$

$$E_0 \left[\max_{k \leq n} |A_{r,k}(\theta, \cdot)|^2 \right] (\omega) \leq 4\alpha^2 + 4 \sum_{j=0}^{n-1} (T^j + E_0 T^j) |(E_0 S_r(\theta)) I_{[|E_0 S_r(\theta)| > \alpha]}|^2 (\omega). \quad (6.28)$$

2. For all $\omega \in \Omega$

$$\int_0^{2\pi} E_0 \left[\max_{k \leq n} |B_{r,k}(\theta, \cdot)|^2 \right] (\omega) d\lambda(\theta) \leq C \sum_{j=2}^{n-1} E_0 |E_{j-1} X_{j+r} - E_0 X_{j+r}|^2 (\omega). \quad (6.29)$$

Proof of Lemma 11: We will prove (6.28) using a truncation argument: let U_α be the (non-linear) operator given by $U_\alpha Y := Y I_{|Y| \geq \alpha}$, and fix the version of E_0 given by $E_0 X(\omega) = E^\omega X$ ($X \in L_{\mathbb{P}}^1$), then for all $\omega \in \Omega$

$$\begin{aligned} \max_{k \leq n} |A_{r,k}(\theta, \cdot)|(\omega) &= \max_{k \leq n} |(Id - E_0)(T^{k-1} E_0 S_r(\theta))|^2(\omega) \leq \\ &4\alpha^2 + 2 \max_{k \leq n} |(Id - E_0) T^{k-1} U_\alpha(E_0 S_r(\theta))|^2(\omega) \leq \\ &4(\alpha^2 + \sum_{j=0}^{n-1} T^j |U_\alpha(E_0 S_r(\theta))|^2(\omega) + \sum_{j=0}^{N-1} E_0 T^j |U_\alpha(E_0 S_r(\theta))|^2(\omega)), \end{aligned}$$

where we used Jensen's inequality. This clearly implies (6.28).

Let us now prove (6.29): by Theorem 2.3 there exists a constant C such that

$$\begin{aligned} \int_0^{2\pi} \max_{k \leq n} |B_{r,k}(\theta, z)|^2 d\lambda(\theta) &\leq C \int_0^{2\pi} \left| \sum_{j=2}^{n-1} (T^j E_{-1} X_r(z) - E_0 T^j E_{-1} X_r(z)) e^{ij\theta} \right|^2 d\lambda(\theta) = \\ &C \sum_{j=2}^{n-1} |E_{j-1} X_{j+r}(z) - E_0 X_{j+r}(z)|^2. \end{aligned}$$

The conclusion follows at once by integrating with respect to E^ω over these inequalities and using Tonelli's theorem. \square

22.1 Proof of Theorem 17.1

Under the hypothesis of Theorem 17.1, if we can prove that there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}\Omega_0 = 1$ such that for all $\omega \in \Omega_0$, (6.14) holds, then, combining this with the proof of Theorem 15.1 (see Section 21) and Lemma 8, the conclusion given in Theorem 17.1 will hold as well.

Let us do so: by Lemma 10, it is sufficient to prove that there exists Ω_0 with $\mathbb{P}\Omega_0 = 1$ such that if for $(k, r, \theta) \in \mathbb{N} \times \mathbb{N}^* \times [0, 2\pi]$ we replace $Z_{r,k}(\theta, \omega) := A_{r,k}(\theta, \omega)$ or $Z_{r,k}(\theta, \omega) := B_{r,k}(\theta, \omega)$, then

$$\lim_r \limsup_n \int_0^{2\pi} E_0 \left[\frac{1}{n} \max_{1 \leq k \leq n} |Z_{r,k}(\theta, \cdot)|^2 \right] (\omega) d\lambda(\theta) = 0. \quad (6.30)$$

for all $\omega \in \Omega_0$.

Proof of (6.30) with $Z_{r,k}(\theta, \omega) := A_{r,k}(\theta, \omega)$: if we fix the version of E_0 given by $E_0 X(\omega) = E^\omega X$ ($X \in L^1_{\mathbb{P}}$) then it is clear that for any $\omega \in \Omega$

$$|E_0 S_r(\theta) I_{[|E_0 S_r(\theta)| > \alpha]}|(\omega) \leq \left| \left(\sum_{j=0}^{r-1} E_0 |X_j| \right) I_{[\sum_{j=0}^{r-1} E_0 |X_j| > \alpha]} \right|(\omega), \quad (6.31)$$

and it follows by an application of Theorem 3.2 (ergodic case), combined with Corollary 4.2 and (6.28) (fixing first $\alpha > 0$ so that the expectation of the random variable at the right in (6.31) is less than any fixed $\eta > 0$), that

$$\lim_n E_0 \left[\frac{1}{n} \max_{1 \leq k \leq n} |A_{r,k}(\theta, \cdot)|^2 \right] = 0 \quad \mathbb{P}-a.s. \quad (6.32)$$

Note that here the (probability one) set $\Omega_{0,1}$ of convergence does not depend on θ and, even more, the convergence is uniform in θ for any fixed $\omega \in \Omega_{0,1}$. It follows that for every $\omega \in \Omega_{0,1}$

$$\begin{aligned} \limsup_n \int_0^{2\pi} E_0 \left[\frac{1}{n} \max_{1 \leq k \leq n} |A_{r,k}(\theta, \cdot)|^2 \right] (\omega) d\lambda(\theta) &\leq \\ \int_0^{2\pi} \limsup_n E_0 \left[\frac{1}{n} \max_{1 \leq k \leq n} |A_{r,k}(\theta, \cdot)|^2 \right] (\omega) d\lambda(\theta) &= 0 \end{aligned}$$

as desired.

Proof of (6.30) with $Z_{r,n}(\theta, \cdot) := B_{r,n}(\theta, \cdot)$: again, fix the version of E_0 given by $E_0 X(\omega) = E^\omega X$. We depart from (6.29) and note that, if for every $j \in \mathbb{Z}$, $X_{-\infty,j} := X_j - E_{-\infty} X_j$ then, by (1.59)

$$\begin{aligned} \sum_{k=2}^{n-1} E_0 |(E_{k-1} - E_0) X_{k+r}|^2 &= \sum_{k=2}^{n-1} E_0 |(E_{k-1} - E_0) X_{-\infty, k+r}|^2 = \\ \sum_{k=2}^{n-1} E_0 T^{k-1} |(E_0 - E_{-k+1}) X_{-\infty, r+1}|^2 &= \sum_{k=1}^{n-2} (E_0 T^k |E_0 X_{-\infty, r+1}|^2 - |E_0 X_{-\infty, k+r+1}|^2) \leq \\ \sum_{k=1}^{n-2} E_0 T^k |E_0 X_{-\infty, r+1}|^2 & \end{aligned}$$

\mathbb{P} -a.s. It follows from (6.29) and Corollary 4.2 that

$$\limsup_{n \rightarrow \infty} \int_0^{2\pi} E_0 \left[\frac{1}{n} \max_{1 \leq k \leq n} |B_{r,k}(\theta, \cdot)|^2 \right] (\omega) d\lambda(\theta) \leq C \|E_0 X_{-\infty, r+1}\|_{\mathbb{P}, 2}^2 = C \|E_{-(r+1)} X_{-\infty, 0}\|_{\mathbb{P}, 2}^2 \quad (6.33)$$

\mathbb{P} -a.s. over a set $\Omega_{0,2,r}$ independent of θ and therefore, by the regularity condition (1.58), (see also (1.60))

$$\lim_r \limsup_n \frac{1}{n} \int_0^{2\pi} E_0 \left[\max_{1 \leq k \leq n} |B_{r,k}(\theta, \cdot)|^2 \right] d\lambda(\theta) = 0$$

for all $\omega \in \Omega_{0,2} := \cap_{r \in \mathbb{N}} \Omega_{0,2,r}$.

To conclude, take $\Omega_0 := \Omega_{0,1} \cap \Omega_{0,2}$. □

22.2 Proof of Theorem 17.2

Let us start by recalling the following (Doob's) maximal inequality ([43], p.53): *if $p > 1$ is given and $(M_k)_{k \in \mathbb{N}^*}$ is a positive submartingale in L_μ^p then*

$$\|M_n\|_{p,\mu} \leq \left\| \max_{0 \leq k \leq n} M_k \right\|_{p,\mu} \leq \frac{p}{p-1} \|M_n\|_{p,\mu}. \quad (6.34)$$

A combination of Doob's maximal inequality (6.34) with Corollary 11.2 gives the following result.

Lemma 12. *With the notation and conventions in page 89, if $(M_k)_{k \in \mathbb{N}^*}$ is a $(\mathcal{F}_{k-1})_{k \in \mathbb{N}^*}$ -adapted martingale in $L_{\mathbb{P}}^2$ then*

$$E_0 \left[\max_{0 \leq k \leq n} |M_k|^2 \right] \leq 4 E_0 |M_n|^2, \quad \mathbb{P}\text{-a.s.} \quad (6.35)$$

To prove Theorem 17.2 we will need some additional estimates which will allow us to exploit the structure brought by (4.22).

Lemma 13. *In the context of Theorem 17.2, and under the notation on page 89, consider the random variables $B_{n,r}(\theta, \cdot)$ given by (6.27). Then for all $(r, n, \theta) \in \mathbb{N} \times \mathbb{N}^* \times [0, 2\pi)$,*

$$\begin{aligned} & |1 - e^{i\theta}| \left(E_0 \left[\max_{k \leq n} |B_{r,k}(\theta, \cdot)|^2 \right] \right)^{\frac{1}{2}} \leq \\ & 2 \sum_{k=1}^{n-4} \left(\sum_{j=1}^{n-2} E_0 T^j |\mathcal{P}_0(X_{k+r+1} - X_{k+r})|^2 \right)^{\frac{1}{2}} + (E_0 |Y(n, r, \theta)|^2)^{1/2} \end{aligned} \quad (6.36)$$

\mathbb{P} -a.s., where the residual $Y(n, r, \theta)$ is such that, under (1.58):

$$\lim_r \limsup_n \frac{1}{n} E_0 |Y(n, r, \theta)|^2 = 0, \quad \mathbb{P}\text{-a.s.} \quad (6.37)$$

Proof: We start by computing

$$\begin{aligned}
(1 - e^{i\theta})B_{n,r}(\theta) &= (TE_{-1}X_r - E_0TE_{-1}X_r)e^{i\theta} \\
&\quad - (TE_{-1}X_r - E_0TE_{-1}X_r)e^{2i\theta} + (T^2E_{-1}X_r - E_0T^2E_{-1}X_r)e^{2i\theta} \\
&\quad \vdots \\
&\quad - (T^{n-2}E_{-1}X_r - E_0T^{n-2}E_{-1}X_r)e^{(n-1)i\theta} + (T^{n-1}E_{-1}X_r - E_0T^{n-1}E_{-1}X_r)e^{(n-1)i\theta} \\
&\quad - (T^{n-1}E_{-1}X_r - E_0T^{n-1}E_{-1}X_r)e^{in\theta} = \\
&\quad - e^{i\theta} \sum_{k=0}^{n-2} (T^k(E_{-1}X_r - TE_{-1}X_r) - E_0T^k(E_{-1}X_r - TE_{-1}X_r))e^{ik\theta} \\
&\quad - (T^{n-1}E_{-1}X_r - E_0T^{n-1}E_{-1}X_r)e^{in\theta}. \tag{6.38}
\end{aligned}$$

Let us stop now to make the following digression: assume that $Y_0 \in L^2_{\mathbb{P}}$ is \mathcal{F}_0 -measurable and let $Y_j := T^j Y_0$ ($j \in \mathbb{Z}$) and

$$S(Y_0, n, \theta) := \sum_{k=0}^{n-1} Y_k e^{ik\theta}$$

the n -th discrete Fourier transform of $(Y_j)_{j \in \mathbb{Z}}$.

Such Y_0 admits the decomposition (see (1.67) and (1.47))

$$Y_0 = \sum_{l=0}^{\infty} \mathcal{P}_{-l} Y_0 + E_{-\infty} Y_0 = \sum_{l=0}^{\infty} T^{-l} \mathcal{P}_0 Y_l + E_{-\infty} Y_0.$$

Since these series are convergent in the $L^2_{\mathbb{P}}$ -sense, it follows that

$$\begin{aligned}
E_0[S(Y_0, n, \theta)] &= \sum_{k=0}^{n-1} \left(\sum_{l=0}^{\infty} E_0 T^k \mathcal{P}_{-l} Y_0 + E_{-\infty} T^k Y_0 \right) e^{ik\theta} = \\
&\sum_{k=0}^{n-1} \left(\sum_{l=k}^{\infty} E_0 T^k \mathcal{P}_{-l} Y_0 + E_{-\infty} T^k Y_0 \right) e^{ik\theta} = \sum_{k=0}^{n-1} \left(\sum_{l=k}^{\infty} T^k \mathcal{P}_{-l} Y_0 + E_{-\infty} T^k Y_0 \right) e^{ik\theta},
\end{aligned}$$

and it follows that

$$\begin{aligned}
(Id - E_0)S(Y_0, n, \theta) &= \sum_{k=0}^{n-1} T^k \left(\sum_{l=0}^{k-1} \mathcal{P}_{-l} Y_0 \right) e^{ik\theta} = \sum_{k=0}^{n-1} \sum_{l=0}^{k-1} T^{k-l} \mathcal{P}_0 Y_l e^{ik\theta} = \\
&\sum_{k=0}^{n-1} \sum_{j=1}^k T^j \mathcal{P}_0 Y_{k-j} e^{ik\theta} = \sum_{k=0}^{n-2} \sum_{j=1}^{n-k-1} T^j \mathcal{P}_0 Y_k e^{i(k+j)\theta}. \tag{6.39}
\end{aligned}$$

To continue towards the proof of (6.36), apply (6.39) with $Y_0 = (Id - T)E_{-1}X_r$ (so that $\mathcal{P}_0 Y_0 = -\mathcal{P}_0 X_{r+1}$ and $\mathcal{P}_0 Y_k = -\mathcal{P}_0(X_{r+k+1} - X_{r+k})$ for $k \geq 1$) to arrive at the identity

$$\begin{aligned} (1 - e^{i\theta})B_{r,n}(\theta) &= -(T^{n-1}E_{-1}X_r - E_0T^{n-1}E_{-1}X_r)e^{in\theta} + e^{i\theta} \sum_{j=1}^{n-1} T^j \mathcal{P}_0 X_{r+1} e^{ij\theta} \\ &\quad + e^{i\theta} \sum_{k=1}^{n-4} \sum_{j=1}^{n-k-1} T^j \mathcal{P}_0 (X_{r+k+1} - X_{r+k}) e^{i(k+j)\theta} \end{aligned}$$

so that, for a fixed $n \geq 4$

$$\max_{0 \leq k \leq n} |(1 - e^{i\theta})B_{k,r}(\theta)| \leq \sum_{j=1}^{n-4} \max_{1 \leq k \leq n-2} \left| \sum_{l=1}^n T^l \mathcal{P}_0 (X_{r+j+1} - X_{r+j}) e^{il\theta} \right| + Y(n, r, \theta) \quad (6.40)$$

where

$$Y(n, r, \theta) := \max_{1 \leq k \leq n} |T^{k-1}E_{-1}X_r - E_0T^{k-1}E_{-1}X_r| + \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k-1} T^j \mathcal{P}_0 X_{r+1} e^{ij\theta} \right|$$

(to see that the “max” can be taken over $n \geq 1$ note that $B_{r,0}(\theta, \cdot) = 0$).

We will prove (6.37) now. To do so we notice that by orthogonality under E_0 , Jensen’s inequality, and Doob’s maximal inequality (6.35),

$$\begin{aligned} E_0(Y(n, r, \theta))^2 &\leq 8 \sum_{j=0}^{n-1} E_0 T^j |E_{-1}X_r|^2 + 8E_0 \left[\sum_{j=1}^{n-1} T^j \mathcal{P}_0 X_{r+1} e^{ij\theta} \right]^2 \\ &= 8 \sum_{j=0}^{n-1} E_0 T^j |E_{-1}X_r|^2 + 8 \sum_{j=1}^{n-1} E_0 T^j |\mathcal{P}_0 X_{r+1}|^2 \end{aligned}$$

\mathbb{P} -a.s. and we use Corollary 4.2 to conclude that

$$\limsup_n \frac{1}{n} E_0(Y(n, r, \theta))^2 \leq 8 \|E_{-(r+1)}X_0\|_{\mathbb{P},2}^2 + 8 \|\mathcal{P}_{-(r+1)}X_0\|_{\mathbb{P},2}^2 \leq 16 \|E_{-(r+1)}X_0\|_{\mathbb{P},2}^2$$

\mathbb{P} -a.s. Using this (6.37) follows clearly from (1.58).

To finish the proof of (6.36) we appeal to (6.40) and we note that, by the conditional Minkowski’s inequality, orthogonality and Doob’s maximal inequality (6.35):

$$\begin{aligned} (E_0 \left[\sum_{j=1}^{n-4} \max_{1 \leq k \leq n-2} \left| \sum_{l=1}^k T^l \mathcal{P}_0 (X_{r+j+1} - X_{r+j}) e^{il\theta} \right|^2 \right])^{1/2} &\leq \\ \sum_{j=1}^{n-4} (E_0 \left[\max_{1 \leq k \leq n-2} \left| \sum_{l=1}^k T^l \mathcal{P}_0 (X_{r+j+1} - X_{r+j}) e^{il\theta} \right|^2 \right])^{1/2} &\leq \end{aligned}$$

$$2 \sum_{j=1}^{n-4} (E_0 [|\sum_{l=1}^{n-2} T^l \mathcal{P}_0(X_{r+j+1} - X_{r+j}) e^{il\theta}|^2])^{1/2} =$$

$$2 \sum_{j=1}^{n-4} (\sum_{l=1}^{n-2} E_0 T^l |\mathcal{P}_0(X_{r+j+1} - X_{r+j})|^2)^{1/2}$$

\mathbb{P} -a.s. □

The following lemma completes the box of tools needed to complete our proof of Theorem 17.2.

Lemma 14. *Under the condition (4.22), the series*

$$\sum_{j=1}^{\infty} (\sup_n \frac{1}{n} \sum_{l=1}^n E_0 T^l |\mathcal{P}_0(X_{j+r+1} - X_{j+r})|^2)^{\frac{1}{2}} \quad (6.41)$$

converges \mathbb{P} -a.s.

Proof: Remember that $L_{\mu}^{2,\infty}$ is a Banach space with the topology given by the norm $|||\cdot|||_{\mu,2}$ (see Section 3.3). Thus by the inequality (1.37) the desired conclusion follows at once taking $Q = E_0 T$ (see also (1.48)), $Y_j = \mathcal{P}_0(X_{j+r+1} - X_{j+r})$, and using condition (4.22). □

Proof of Theorem 17.2: Our goal is to verify that, under the hypotheses of Theorem 17.2, the approximation (6.7) in Lemma 7 holds. Note that, by (6.22) and (6.32), it suffices to prove that, under (4.22),

$$\lim_r \limsup_n E_0 \frac{1}{n} \max_{1 \leq k \leq n} |B_{r,k}(\theta, \cdot)|^2 = 0 \quad \mathbb{P}\text{-a.s.} \quad (6.42)$$

for every $\theta \neq 0$.

Reduction: We start from the following observation: by (1.47) and (1.59), the definition of $B_{r,n}(\theta, \cdot)$ remains unchanged if we replace X_r by $X_{-\infty,r} := X_r - E_{-\infty} X_r$. Thus we can assume, without loss of generality, that $(X_k)_{k \in \mathbb{Z}}$ is regular (Definition 5.4, see also (1.60)).

Proof of (6.42) under the assumption of regularity (see the reduction above): it follows from (6.36) that

$$\left(\frac{1}{n} E_0 \left[\max_{k \leq n} |B_{r,k}(\theta, \cdot)|^2 \right] \right)^{\frac{1}{2}} \leq$$

$$|1 - e^{i\theta}|^{-1} \left(2 \sum_{j=1}^{n-4} \left(\frac{1}{n} \sum_{l=1}^n E_0 T^l |\mathcal{P}_0(X_{j+r+1} - X_{j+r})|^2 \right)^{\frac{1}{2}} + \left(\frac{1}{n} E_0 (Y(n, r, \theta))^2 \right)^{1/2} \right),$$

\mathbb{P} -a.s.

So from (6.37), Lemma 14, the dominated convergence theorem, and Corollary 4.2 we get that

$$\limsup_{n \rightarrow \infty} \left(\frac{1}{n} E_0 \left[\max_{k \leq n} |B_{r,k}(\theta, \cdot)|^2 \right] \right)^{\frac{1}{2}} \leq 2|1 - e^{i\theta}|^{-1} \sum_{j=r+1}^{\infty} \|\mathcal{P}_0(X_{j+1} - X_j)\|_{\mathbb{P},2} + o_r(1) \quad (6.43)$$

\mathbb{P} -a.s. and therefore, by condition (4.22) again³

$$\lim_r \limsup_n \left(\frac{1}{n} E_0 \left[\max_{k \leq n} |B_{r,k}(\theta, \cdot)|^2 \right] \right)^{\frac{1}{2}} = 0 \quad (6.44)$$

\mathbb{P} - a.s., as desired. \square

Remark 22.1. The set Ω_0 of convergence in the last statement can be chosen independent of θ (this requires some care, but the general strategy is to use the representation $\omega \mapsto E^\omega X$ of $E[X|\mathcal{F}_0]$ along all of the arguments). It is not clear, on the other side, whether the convergence is uniform in θ for a fixed $\omega \in \Omega_0$ (due to the factor $|1 - e^{i\theta}|^{-1}$). Contrast this with the statement following (6.32).

22.3 A Note on Theorem 10.1

The proofs presented along this chapter can be carried out using the following (more restrictive) classical version of Theorem 10.1 (see Theorem 3.2 in [10] for a proof).

Theorem 22.1 (Transport Theorem). *Let (S, d) be a complete and separable metric space. Assume that for all natural numbers r, n , $X_{r,n}$ and X_n are random elements of S defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that $X_{r,n} \Rightarrow_n Z_r$, and that $Z_r \Rightarrow_r X$. Then the hypothesis*

$$\lim_r \limsup_n \mathbb{P}[d(X_{r,n}, X_n) \geq \epsilon] = 0 \quad \text{for all } \epsilon > 0, \quad (6.45)$$

implies that $X_n \Rightarrow_n X$.

The adaptation of the arguments above to the further restriction imposed by this theorem poses no serious difficulty: it suffices to see that the martingales $D_{r,0}$ converge in the appropriate way, as $r \rightarrow \infty$, for each one of the theorems proved along this section, and in particular to use the results presented in Chapter 5 to deduce the asymptotic distributions associated to the processes generated via the martingale differences $D_0 = \lim_r D_{r,0}$. The details are left to the reader.

³Note that (4.22) was used already when applying Lemma 14.

Chapter 7

Proofs Related to the Random Centering

In this chapter we present proofs of the results stated, but not proved, in Section 16.

Section 23, devoted to the proof of Theorem 16.1, addresses the problem of the meaning of convergence under \mathbb{P}_ω , for a fixed (and appropriately chosen) ω , of a stochastic process $(Y_n)_n$ for which $Y_n - E_0 Y_n$ converges in the quenched sense. The main novelty is Proposition 23.1, a relaxed form of Theorem 16.1 from which this result follows easily.

Section 24 can be considered as pertaining to an exposition on the general theory of quenched convergence, but we present it in this part of the monograph because, first, it can be considered as a relatively straightforward specialization of the results presented in Chapter 4, and second, the exposition is written with the purpose of giving a precise interpretation of the processes there considered (adapted linear processes) under the light of the hypotheses in theorems 5.5 and 15.1, paving the way to the arguments present in Section 25 towards the proof of Theorem 16.3. The main result in this section is Proposition 24.1.

Section 25 presents the proof of Theorem 16.3. We start with two approximation lemmas (lemmas 16 and 17) and then proceed to give an instance of the process announced by Theorem 16.3 by following a construction due to Volný and Woodroffe.

Finally, in Section 26, we present the proofs of theorems 16.5 and 16.6. Besides presenting these proofs, this section aims to illustrate how the techniques involved in the proofs of some previous results in this monograph can (and should) be used to expand the family of theorems on quenched asymptotics for the Discrete Fourier Transforms of dependent sequences by combining the estimates present in the existing literature for non-rotated partial sums with the martingale limit theorems developed along this work.

The notation is that introduced at the beginning and on page 89.

23 Proof of Theorem 16.1

To begin with, suppose that we know that the integrable process $(Y_n)_{n \in \mathbb{N}}$ is such that that $Y_n - E_0 Y_n \Rightarrow Y$ in the quenched sense. Thus (Proposition 13.1) there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}\Omega_0 = 1$ such that for every $\omega \in \Omega_0$, $Y_n - E_0 Y_n \Rightarrow Y$ under \mathbb{P}_ω .

Question: *What are the possible limit laws for $(Y_n)_{n \in \mathbb{N}}$ under \mathbb{P}_ω for a fixed ω ?*

To answer this question we depart from the following auxiliary result. Remember that, for a metric space S , $\mathbf{C}^b(S)$ denotes the space of functions $h : S \rightarrow \mathbb{R}$ that are continuous and bounded.

Lemma 15. *Under the setting introduced in page 89, if X is \mathcal{F} -measurable and Z is \mathcal{F}_0 -measurable¹, there exists $\Omega(X, Z) \subset \Omega$ with $\mathbb{P}(\Omega(X, Z)) = 1$ such that, for every $h \in \mathbf{C}^b(\mathbb{C}^2)$*

$$E^\omega[h(X, Z)] = E^\omega[h(X, Z(\omega))] \quad (7.1)$$

for all $\omega \in \Omega(X, Z)$.

Proof: Let $A \in \mathcal{F}_0$ be given, and assume first that $Z = I_A$. Then, clearly

$$h(X, Z) = h(X, 1)I_A + h(X, 0)I_{\Omega \setminus A}$$

and therefore

$$E_0 h(X, Z) = E_0[h(X, 1)]I_A + E_0[h(X, 0)]I_{\Omega \setminus A}$$

\mathbb{P} -a.s., which implies, via the representation $E_0 Y(\omega) = E^\omega Y$ ($Y \in L^1_{\mathbb{P}}$), that there exists $\Omega(h, X, Z)$ with $\mathbb{P}\Omega(h, X, Z) = 1$ such that (7.1) holds for every $\omega \in \Omega(h, X, Z)$. This argument can be easily extended to the case of simple functions $Z = \sum_{k=1}^n a_k I_{A_k}$ with $A_k \in \mathcal{F}_0$ ($k = 1, \dots, n$).

Now assume that Z is an arbitrary \mathcal{F}_0 -measurable function, let $(Z_n)_{n \in \mathbb{N}}$ be a sequence of simple functions with $\lim_n Z_n(\omega) = Z(\omega)$ ([11], p.254), and let

$$\Omega(h, X, Z) := \cap_{n \in \mathbb{N}} \Omega(h, X, Z_n).$$

Clearly $\mathbb{P}\Omega(h, X, Z) = 1$, and the dominated convergence theorem, together with the definition of $\Omega(h, X, Z)$ and the continuity of the bounded function h imply that for every $\omega \in \Omega(h, X, Z)$

$$E^\omega(h(X, Z)) = \lim_n E^\omega h(X, Z_n) = \lim_n E^\omega h(X, Z_n(\omega)) = E^\omega h(X, Z(\omega)).$$

Finally, let $(h_n)_{n \in \mathbb{Z}}$ be a family of Urysohn functions as in the statement 2. of Theorem 6.1, and let

$$\Omega(X, Z) := \cap_{n \in \mathbb{N}} \Omega(h_n, X, Z).$$

Again, $\mathbb{P}\Omega(X, Z) = 1$, and by Theorem 6.1 (replacing X_n by (X, Z) for all $n \in \mathbb{N}$ and X by $(X, Z(\omega))$) and the definition of $\Omega(X, Z)$, for every $\omega \in \Omega(X, Z)$ and every $h \in \mathbf{C}^b(\mathbb{C}^2)$, $E^\omega h(X, Z) = E^\omega h(X, Z(\omega))$. \square

This lemma, in combination with Proposition 8.3 gives the following result.

¹These are *not* \mathbb{P} -equivalence classes, but actual “versions” of X and Z .

Proposition 23.1 (Possible Limit Laws for a Fixed Starting Point). *With the notation of Lemma 15, assume that $(Z_n)_n$ is a sequence of functions in $L^1_{\mathbb{P}}$ such that $Z_n - E_0 Z_n \Rightarrow_n Y$ under \mathbb{P}_ω for all $\omega \in \Omega_{0,1}$ ($\Omega_{0,1} \subset \Omega$ is any given set, not even assumed measurable), and let $\Omega_{0,2} := \cap_n \Omega(Z_n, E_0 Z_n)$, where $\Omega(Z_n, E_0 Z_n)$ is the set specified in the conclusion of Lemma 15². Then, given $\omega \in \Omega_0 := \Omega_{0,1} \cap \Omega_{0,2}$, $Z_n \Rightarrow Z_\omega$ under \mathbb{P}_ω if and only if $L(\omega) = \lim_{n \rightarrow \infty} E_0 Z_n(\omega)$ exists, in which case*

$$Z_\omega = Y + L(\omega) \quad (7.2)$$

(in distribution).

Proof: Given $\omega \in \Omega_0$ and any bounded and continuous function $h : \mathbb{C} \rightarrow \mathbb{R}$, the hypotheses imply that

$$E^\omega h(Z_n - E_0 Z_n(\omega)) = E^\omega h(Z_n - E_0 Z_n) \rightarrow Eh(Y),$$

as $n \rightarrow \infty$, so that $Z_n - E_0 Z_n(\omega) \Rightarrow Y$ under \mathbb{P}_ω (Portmanteau's Theorem). From $Z_n = Z_n - E_0 Z_n(\omega) + E_0 Z_n(\omega)$ the conclusion follows via Proposition 8.3 in page 53. \square

We can proceed now to the proof of Theorem 16.1

Proof of Theorem 16.1: We appeal to Proposition 23.1, replacing Z_n by $Z_n(\theta)$ and taking

$$\Omega_{\theta,1} := \{\omega \in \Omega : Y_n(\theta) = Z_n(\theta) - E_0 Z_n(\theta) \Rightarrow Y(\theta) \text{ under } \mathbb{P}_\omega\}$$

and $\Omega_{\theta,2} = \cap_{n \in \mathbb{N}} \Omega(Z_n(\theta), E_0 Z_n(\theta))$. This gives that, for any $\omega \in \Omega_\theta := \Omega_{\theta,1} \cap \Omega_{\theta,2}$ fixed, there exists $L_\theta(\omega) := \lim_n E_0 Z_n(\theta, \omega)$ and therefore, if $Z_n(\theta) \Rightarrow_n Z_\omega(\theta)$ under \mathbb{P}_ω , $Z_\omega(\theta) = Y(\theta) + L_\theta(\omega)$ (in distribution) under \mathbb{P}_ω . With this observation, the conclusion follows easily from Proposition 8.3 and the fact that $\mathbb{P}\Omega_\theta = 1$. \square

As explained at the end of Section 16.1 (see the “General Comments”), it follows from Corollary 16.2 that if we can provide an example of a regular process $(X_n)_n$ (Definition 5.4) satisfying the hypothesis of Theorem 15.1 for which

$$\mathbb{P} \left(\limsup_{n \rightarrow \infty} \left| \frac{1}{\sqrt{n}} E_0 S_n(\theta) \right| > 0 \right) > 0 \text{ for } \theta \text{ in a set } I' \text{ with } \lambda(I') > 0 \quad (7.3)$$

we will prove, in particular, the necessity of the random centering “ $-E_0 Z_n(\theta)$ ” for a nonnegligible subset of I (namely $I \cap I'$).

In their paper [45], Volný and Woodroffe provide an example of a sequence $(X_n)_n$ for which a quenched CLT holds for $(Y_n(0))_n$ but not for $(Z_n(0))_n$. We will adapt their construction to give an example satisfying (4.11) for every $\theta \in [0, 2\pi)$ (this clearly implies (7.3)). By the discussion at the end of Section 16.1 again, this will make the proof of Theorem 16.3.

The main novelty adapting the example in [45], which arises from a careful construction of a sequence $(a_n)_{n \in \mathbb{N}}$ of nonnegative coefficients of a linear process is that, in order to

²Of course, here we are implicitly fixing versions of Z_n and $E_0 Z_n$. We will leave these details to the reader.

guarantee the validity of the “inductive step” defining a_{n+1} from a_1, \dots, a_n , one needs to prove that a certain type of convergence is uniform in θ (see Lemma 16 below). While it would be sufficient to prove this uniform convergence for θ in (for instance) an open subinterval I' of $[0, 2\pi)$ in order to construct a valid example, a compactness argument allows us to do it for $I' = [0, 2\pi)$.

24 Theorem 15.1 for Adapted Linear Processes

Let $(a_k)_{k \in \mathbb{N}} \in l^2(\mathbb{N})$ be given, thus $a_k \in \mathbb{C}$ for every $k \in \mathbb{N}$, and $\sum_k |a_k|^2 < \infty$. As explained in Section 2.2, Carleson’s theorem (Theorem 2.1) guarantees the convergence, for λ -a.e θ , of the series

$$\sum_{j \geq 0} a_j e^{ij\theta}$$

and therefore there exists a well defined (λ -equivalence class of) function(s) $f : [0, 2\pi) \rightarrow \mathbb{C}$ given by

$$\theta \mapsto f(\theta) = \lim_n \sum_{j=0}^{n-1} a_j e^{ij\theta} \quad (7.4)$$

(see Definition 2.2) and $f(\theta)$, thus defined, is a 2π -periodic function, square integrable over $[0, 2\pi)$, and satisfying $\hat{f}(n) = a_n$ for every $n \in \mathbb{N}$, where \hat{f} denotes the Fourier transform (1.9).

For every $k \in \mathbb{Z}$, denote by

$$f_k(\theta) := \sum_{j=0}^{k-1} a_j e^{ij\theta} \quad (7.5)$$

(thus $f_k = 0$ if $k \leq 0$), and consider the setting explained along Example 5 in page 64. If we regard $(a_k)_{k \in \mathbb{N}}$ as an element of $l^2(\mathbb{Z})$ with $a_k = 0$ for $k < 0$ then, as explained in Example 1, the process $(X_k)_{k \in \mathbb{Z}}$ given by (1.45) is $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ -adapted. Since in this setting $(x_k)_{k \in \mathbb{Z}}$, the sequence of coordinate functions, is i.i.d., Kolmogorov’s zero-one law ([11], Theorem 22.3) implies that for every $k \in \mathbb{Z}$

$$E[X_k | \mathcal{F}_{-\infty}] = E[X_k] = 0$$

and it follows that $(X_k)_{k \in \mathbb{Z}}$ is regular (see 1.60).

In conclusion, these processes satisfy the hypotheses of Theorem 15.1 and are regular. This will be the basis for the construction of the example stated by Theorem 16.3.

To begin our discussions, start by noting that, in the context just explained, we have the following two expressions for the Fourier Transforms $S_n(\theta)$ (Definition 2.6) of $(X_k)_{k \in \mathbb{Z}}$:

$$S_n(\theta) = \sum_{j=-\infty}^{n-1} (f_{-j+n} - f_{-j})(\theta) x_j e^{ij\theta}, \quad (7.6)$$

$$S_n(\theta) = \sum_{k=0}^{\infty} a_k \sum_{j=0}^{n-1} e^{ij\theta} x_{j-k}. \quad (7.7)$$

in the $L^2_{\mathbb{P}}$ -sense³. Now let us denote, for all $k \in \mathbb{Z}$,

$$\zeta_{-k}(\theta) := \sum_{j=0}^k e^{-ij\theta} x_{-j} \quad (7.8)$$

(note that $\zeta_{-k} = 0$ if $k < 0$). Then from (7.6) and (7.7) the following two equalities follow respectively:

$$E_0 S_n(\theta) = \sum_{j \in \mathbb{N}} x_{-j} (f_{j+n} - f_j)(\theta) e^{-ij\theta}, \quad (7.9)$$

$$E_0 S_n(\theta) = \sum_{j \in \mathbb{N}} a_j (\zeta_{-j} - \zeta_{-j+n})(\theta) e^{ij\theta}. \quad (7.10)$$

In particular

$$\begin{aligned} E_0 |S_n(\theta) - E_0 S_n(\theta)|^2 &= E_0 \left| \sum_{j=1}^{n-1} e^{ij\theta} x_j f_{-j+n}(\theta) \right|^2 = \\ &= \|x_0\|_{\mathbb{P},2}^2 \sum_{j=1}^{n-1} |f_{n-j}(\theta)|^2 = \|x_0\|_{\mathbb{P},2}^2 \sum_{j=1}^{n-1} |f_j(\theta)|^2, \end{aligned}$$

so that, by Theorem 15.1, there exists $I \subset (0, 2\pi)$ with $\lambda(I) = 1$ such that for every $\theta \in I$

$$\lim_n \frac{E_0 |S_n(\theta) - E_0 S_n(\theta)|^2}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n-1} \|x_0\|_{\mathbb{P},2}^2 |f_j(\theta)|^2 = \|x_0\|_2^2 |f(\theta)|^2.$$

Using this fact, we get the following version of Theorem 15.1:

Proposition 24.1 (Theorem 15.1 for Adapted Linear Processes). *With the notations and the setting in Example 1 and Example 5, and given a $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ -adapted linear process (1.45) (thus $a_k = 0$ if $k < 0$), there exists $I \subset (0, 2\pi)$ such that for every $\theta \in I$, $(S_n(\theta) - E_0 S_n(\theta))/\sqrt{n}$ is asymptotically normally distributed under \mathbb{P}_ω for \mathbb{P} -a.e ω (\mathbb{P}_ω is given by (3.10)), and its asymptotic distribution (under \mathbb{P}_ω) corresponds to a complex-valued normal random variable with independent real and imaginary parts, each with mean zero and variance*

$$\sigma_\theta^2 = \frac{\|x_0\|_{\mathbb{P},2}^2 |f(\theta)|^2}{2},$$

where f is given by (7.4).

³The changes in the order of summation involved can be easily justified in this case. We skip this detail.

25 Proof of Theorem 16.3

We finally address here the construction leading to the proof of Theorem 16.3. The notation along the following arguments is borrowed from the previous sections in this chapter. In particular, ζ_n is defined by (7.8) for every $n \in \mathbb{Z}$, and T is the left shift in $\mathbb{R}^{\mathbb{Z}}$ which, under the setting of Example 5, is weakly mixing (see [42], p.13).

By [19], p.4075 (Section 4.1) applied to the sequence $(\delta_{1j})_{j \in \mathbb{Z}}$ (δ_{ij} denotes the Kronecker δ -function) and the fact that T is weakly mixing, the following law of the iterated logarithm holds⁴: for every $t \in (0, 2\pi) \setminus \{\pi\}$

$$\limsup_{n \rightarrow \infty} \frac{|\zeta_{-n}(\theta)|}{\sqrt{n \log \log n}} = \|x_0\|_{\mathbb{P}, 2}. \quad (7.11)$$

\mathbb{P} -almost surely.

If $\theta = 0$ or $\theta = \pi$, and x_0 is real-valued and symmetric ($\mathbb{P}[x_0 \leq t] = \mathbb{P}[x_0 \geq -t]$), the L.I.L. as stated above holds with $\|x_0\|_{\mathbb{P}, 2}$ replaced by $\sqrt{2}\|x_0\|_{\mathbb{P}, 2}$ ([11], Theorem 9.5). Assume this from now on.

The equality (7.11) clearly implies that

$$\limsup_n \frac{|\zeta_{-n}(\theta)|}{\sqrt{n}} = \infty$$

\mathbb{P} -a.s. The following lemma states that the divergence occurs “at comparable speeds” for every θ .

Lemma 16. *Consider ζ_{-k} as defined by (7.8). Then for every $\lambda \in \mathbb{R}$ and every $0 < \eta \leq 1$ there exists $N \in \mathbb{N}$ satisfying*

$$\mathbb{P} \left(\max_{1 \leq n \leq N} \frac{|\zeta_{-n}(\theta)|}{\sqrt{n}} > \lambda \right) \geq 1 - \eta$$

for all $\theta \in [0, 2\pi)$. In particular

$$\mathbb{P} \left(\max_{1 \leq n \leq m} \frac{|\zeta_{-n}(\theta)|}{\sqrt{n}} \geq \lambda \right) \geq 1 - \eta$$

for all $m \geq N$.

Proof: Fix $\lambda \in \mathbb{R}$ and $0 < \eta \leq 1$. Let⁵ $\theta \in [0, 2\pi]$ and $\epsilon > 0$ be given and define

⁴Note that, for the linear process $(x_n)_{n \in \mathbb{Z}}$ (the coordinate functions), which corresponds to convolution with the sequence $(\delta_{1j})_{j \in \mathbb{Z}} \in l^2(\mathbb{Z})$, the spectral density with respect to Lebesgue measure is the constant function $\|x_0\|_{\mathbb{P}, 2}^2/2\pi$.

⁵We work over the interval $[0, 2\pi]$ (instead of $[0, 2\pi)$). This has no effect for the validity of the conclusion and is assumed in order to take advantage of compactness, as will be clear along the proof.

$$E_{\epsilon,m}(\theta) := \left[\inf_{|\delta| < \epsilon} \left\{ \max_{1 \leq n \leq m} \frac{|\zeta_{-n}(\theta + \delta)|}{\sqrt{n}} \right\} > \lambda \right]$$

and

$$E_m(\theta) := \left[\max_{1 \leq n \leq m} \frac{|\zeta_{-n}(\theta)|}{\sqrt{n}} > \lambda \right].$$

Note that, for fixed m , the sequence of sets $E_{\epsilon,m}(\theta)$ is decreasing with respect to ϵ ($\epsilon_1 < \epsilon_2$ implies that $E_{\epsilon_2,m}(\theta) \subset E_{\epsilon_1,m}(\theta)$), and that the (random) function $\theta \mapsto \max_{1 \leq n \leq m} |\zeta_{-n}(\theta)|/\sqrt{n}$ is continuous for all m . In particular

$$\bigcup_{\epsilon > 0} E_{\epsilon,m}(\theta) = E_m(\theta), \quad (7.12)$$

where the sets in the union increase as ϵ decreases to 0.

Now, there exists a minimal $N(\theta)$ such that $\mathbb{P}(E_{N(\theta)}(\theta)) > 1 - \eta$. To see this note that the family $\{E_k(\theta)\}_{k \geq 0}$ is increasing with k , and its union contains the set

$$[\limsup_n |\zeta_{-n}(\theta)|/\sqrt{n} > \lambda]$$

which has \mathbb{P} -measure 1 by (7.11).

It follows from (7.12) that there exists an ϵ_θ such that

$$\mathbb{P}(E_{\epsilon_\theta, N(\theta)}(\theta)) > 1 - \eta. \quad (7.13)$$

Now, the family of sets $\{(\theta - \epsilon_\theta, \theta + \epsilon_\theta)\}_{\theta \in [0, 2\pi]}$ is an open cover of $[0, 2\pi]$, and therefore it admits an open subcover $\{(\theta_j - \epsilon_j, \theta_j + \epsilon_j)\}_{j=1}^r$ where $\epsilon_j := \epsilon_{\theta_j}$. Let $N = \max\{N(\theta_1), \dots, N(\theta_r)\}$. We claim that for every $\theta \in [0, 2\pi]$

$$\mathbb{P}(E_N(\theta)) > 1 - \eta.$$

Indeed, given $\theta \in [0, 2\pi]$, with $\theta_j - \epsilon_j < \theta < \theta_j + \epsilon_j$,

$$E_N(\theta) \supset E_{N(\theta_j)}(\theta) = \left[\max_{1 \leq n \leq N(\theta_j)} \frac{|\zeta_{-n}(\theta_j + \theta - \theta_j)|}{\sqrt{n}} > \lambda \right] \supset E_{\epsilon_j, N(\theta_j)}(\theta_j),$$

and the conclusion follows from (7.13) and the definition of $E_N(\theta)$. \square

Let us now move to the following observation: if $(n_k)_{k \in \mathbb{N}}$ is a strictly increasing sequence of natural numbers and if $(a_k)_{k \in \mathbb{N}}$ is square summable and satisfies $a_j = 0$ if $j \notin \{n_k\}_k$ then, using (7.10) we have, for every given $k \in \mathbb{N}$,

$$E_0 S_n(\theta) = \sum_{j=0}^k e^{in_j \theta} a_{n_j} (\zeta_{-n_j} - \zeta_{-n_j+n})(\theta) + \sum_{j=k+1}^{\infty} e^{in_j \theta} a_{n_j} (\zeta_{-n_j} - \zeta_{-n_j+n})(\theta) =:$$

$$A_k(n, \theta) + B_k(n, \theta) \quad (7.14)$$

so that

$$\begin{aligned} & \mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} \geq 2^k \right) \geq \\ & \mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|A_k(n, \theta)|}{\sqrt{n}} \geq 2^{k+1} \right) - \mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|B_k(n, \theta)|}{\sqrt{n}} \geq 2^k \right) \geq \\ & \mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|A_k(n, \theta)|}{\sqrt{n}} \geq 2^{k+1} \right) - \mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} |B_k(n, \theta)| \geq 2^k \right). \end{aligned} \quad (7.15)$$

Now, if $n_{k-1} < n \leq n_k$ then, actually

$$A_k(n, \theta) = \sum_{j=0}^{k-1} e^{in_j \theta} a_{n_j} \zeta_{-n_j}(\theta) + e^{in_k \theta} a_{n_k} (\zeta_{-n_k} - \zeta_{-n_k+n})(\theta).$$

The first summand at the right-hand side in this expression is bounded by

$$\sum_{j=1}^{k-1} \sum_{r=0}^{n_j} |a_{n_j}| |\xi_{-r}|$$

and therefore there exists $\gamma_k > 0$ such that

$$\mathbb{P} \left(\left| \sum_{j=0}^{k-1} e^{in_j \theta} a_{n_j} \zeta_{-n_j}(\theta) \right| > \gamma_k \right) \leq \left(\frac{1}{2} \right)^{k+2} \quad (7.16)$$

for all $\theta \in [0, 2\pi]$.

All together (7.14), (7.15) and (7.16) give the following result.

Lemma 17. *Let $(n_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers and let $(a_j)_{j \in \mathbb{N}} \in l^2(\mathbb{N})$ be a square summable sequence of nonnegative numbers with $a_j = 0$ for $j \notin \{n_k\}_k$. Then for every sequence of real numbers $(\gamma_k)_k$ satisfying (7.16) the following inequality holds*

$$\begin{aligned} & \mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} \geq 2^k \right) \geq \mathbb{P} \left(a_{n_k} \max_{n_{k-1} < n \leq n_k} \frac{|(\zeta_{-n_k} - \zeta_{-n_k+n})(\theta)|}{\sqrt{n}} \geq \gamma_k + 2^{k+1} \right) \\ & - \mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} |B_k(n, \theta)| \geq 2^k \right) - \left(\frac{1}{2} \right)^{k+2} \end{aligned} \quad (7.17)$$

for all $\theta \in [0, 2\pi]$.

This completes the set of pieces needed to construct the example stated by Theorem 16.3.

Proof of Theorem 16.3: Following [45], assume that $\|x_0\|_{\mathbb{P},2} = 1$ and let $(n_j)_{j \geq 0}$, $(a_j)_{j \geq 0}$, and $(\gamma_j)_{j \geq 0}$ be defined inductively as follows: $n_0 = 1$, $\gamma_0 = 0$, $a_0 = 0$, $a_1 = \frac{1}{2}$, and given n_0, \dots, n_{k-1} , $a_0, \dots, a_{n_{k-1}}$ and $\gamma_0, \dots, \gamma_{k-1}$, let γ_k be such that

$$\mathbb{P} \left(\left| \sum_{j=1}^{k-1} a_{n_j} e^{in_j \theta} \zeta_{-n_j}(\theta) \right| > \gamma_k \right) \leq \left(\frac{1}{2} \right)^{k+2},$$

(see (7.16)) and let $n_k > n_{k-1}$ be such that

$$\mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|(\zeta_{-n_{k-1}} - \zeta_{-n_{k-1}+n})(\theta)|}{\sqrt{n}} \geq \frac{\gamma_k + 2^{k+1}}{a_{n_{k-1}}} \right) \geq 1 - \left(\frac{1}{2} \right)^{k+1} \quad (7.18)$$

for all $\theta \in [0, 2\pi]$. The choice of n_k is possible according to Lemma 16 ($|(\zeta_{-n_{k-1}} - \zeta_{-n_{k-1}+n})(\theta)|$ and $|\zeta_{-n}(\theta)|$ have the same distribution). Then define $a_{n_k} = \frac{1}{2^k \sqrt{n_{k-1}}}$ and $a_j = 0$ for $n_{k-1} < j < n_k - 1$.

The sequences $(a_j)_{j \geq 0}$ and $(\gamma_k)_k$, thus defined, satisfy the hypotheses of Lemma 17 and therefore, by the estimates (7.17) and (7.18),

$$\mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} \geq 2^k \right) \geq 1 - \left(\frac{1}{2} \right)^{k+2} - \mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} |B_k(n, \theta)| \geq 2^k \right)$$

for all $\theta \in [0, 2\pi]$.

We claim that, under the present conditions,

$$\mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} |B_k(n, \theta)| \geq 2^k \right) \leq \left(\frac{1}{2} \right)^{k+2} \quad (7.19)$$

for $k \geq 3$.

Fix $k \geq 3$ and note that, for fixed θ , $(|\zeta_{-n}(\theta)|)_{n \in \mathbb{N}}$ is an $L^2_{\mathbb{P}}$ submartingale (with respect to $(\mathcal{G}_n)_{n \in \mathbb{Z}}$, where $\mathcal{G}_k = \sigma((x_{-j})_{j \leq k})$) and therefore, by Doob's maximal inequality (6.34):

$$E \left(\max_{k \leq n} |\zeta_{-k}(\theta)| \right) \leq \left\| \max_{k \leq n} |\zeta_{-k}(\theta)| \right\|_{\mathbb{P},2} \leq 2 \|\zeta_{-n}(\theta)\|_2 = 2 \sqrt{n+1}.$$

This gives

$$\begin{aligned} E \left(\max_{n_{k-1} < n \leq n_k} |B_k(n, \theta)| \right) &\leq \sum_{j=k+1}^{\infty} a_{n_j} E \left(\max_{k \leq n_k - n_{k-1}} |\zeta_{-k}(\theta)| \right) \leq \\ &\sum_{j=k+1}^{\infty} \frac{1}{2^{j-1}} \sqrt{\frac{n_k - n_{k-1} + 1}{n_{j-1}}} \leq \frac{1}{2^{k-1}}, \end{aligned}$$

and therefore, by Markov's inequality ([11], p.276, (21.12))

$$\mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} |B_k(n, \theta)| \geq 2^k \right) \leq \frac{1}{2^{2k-1}} \leq \left(\frac{1}{2} \right)^{k+2}$$

as claimed.

To finish the proof we observe that a combination of (7.17), (7.18) and (7.19) gives, under the present choices of $(a_k)_k$ and $(n_k)_k$, that

$$\mathbb{P} \left(\max_{n_{k-1} < n \leq n_k} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} < 2^k \right) \leq \left(\frac{1}{2} \right)^{k+1}$$

so that, by the first Borel-Cantelli Lemma ([11], Theorem 4.3)

$$\max_{n_{k-1} < n \leq n_k} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} \geq 2^k \text{ except for finitely many } k's,$$

\mathbb{P} -a.s. This clearly implies that $\limsup_n |E_0 S_n(\theta)|/\sqrt{n} = \infty$ \mathbb{P} -a.s. □

26 Proof of Theorems 16.5 and 16.6

In this section we address the proofs of theorems 16.5 and 16.6. As the reader may expect at this point, these are just consequences of the fact that the hypotheses in these theorems are sufficient to verify the validity of item 2. in Corollary 16.2 (page 80).

It is important to point out that, for proving Theorem 16.6, we will use again the extensions to the product space described in Definition 3.3 and (1.30). Together with the proofs of Theorem 3.2, Theorem 4.1 and Lemma 9 (“*Step 2.*”), this will serve as a further illustration of how this method of *lifting to the product space* allows us to translate estimates on (non-rotated) partial sums to corresponding results for Discrete Fourier Transforms.

26.1 Proof of Theorem 16.5

We will use the criterion given in Corollary 16.2: we will prove that there exists $I' \subset [0, 2\pi)$ with $\lambda(I') = 1$ such that $(1 - e^{i\theta})E_0 S_n(\theta)/\sqrt{n} \rightarrow_n 0$, \mathbb{P} -a.s for every $\theta \in I'$. The conclusion follows by taking $J = I \cap I'$, where I is the set guaranteed by Theorem 15.1 and using item 2. in Corollary 16.2 (note that, for $\theta \in I$, $e^{i\theta} \neq 1$).

To do so we use an argument similar in spirit to the one leading to the proof of Lemma 13 (the decompositions are way simpler here). Thus note that, if $n \in \mathbb{N}^*$ and $\theta \in [0, 2\pi)$ are given

$$(1 - e^{i\theta})E_0 \frac{S_n(\theta)}{\sqrt{n}} = \frac{E_0(S_n(\theta)) - e^{i\theta} E_0(S_n(\theta))}{\sqrt{n}} =$$

$$\frac{1}{\sqrt{n}}e^{i\theta}X_0 - e^{in\theta}\frac{1}{\sqrt{n}}E_0X_{n-1} + \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1}E_0[X_k - X_{k-1}]e^{ik\theta}. \quad (7.20)$$

We will analyze each term in the last sum separately: the first term in the above expression, $e^{it}X_0/\sqrt{n}$, is trivially convergent to zero for every $\omega \in \Omega$.

Now, the conditional Jensen's inequality gives that:

$$|e^{in\theta}\frac{1}{\sqrt{n}}E_0X_{n-1}|^2 \leq \frac{1}{n}E_0|X_{n-1}|^2 \quad (7.21)$$

\mathbb{P} -a.s., and if we write

$$\frac{1}{n}E_0|X_{n-1}|^2 = \frac{1}{n}\sum_{j=0}^{n-1}E_0T^j|X_0|^2 - \frac{1}{n}\sum_{j=0}^{n-2}E_0T^j|X_0|^2,$$

we see that $E_0|X_{n-1}|^2/n \rightarrow_n 0$, \mathbb{P} -a.s. by Theorem 4.1, and therefore $e^{in\theta}E_0X_{n-1}/\sqrt{n} \rightarrow 0$, \mathbb{P} -a.s. by (7.21).

To prove the convergence of the third term note that, since we are under the assumption (4.14), an argument similar to the one leading to the proof of the \mathbb{P} -a.s convergence of (1.13) for λ -a.e θ (page 17) implies that there exists $I' \subset [0, 2\pi)$ with $\lambda(I') = 1$ such that for every $\theta \in I'$

$$\sum_{k \in \mathbb{N}^*} \frac{E_0(X_k - X_{k-1})}{k^{1/2}} e^{ik\theta}$$

converges \mathbb{P} -a.s, and the Kronecker lemma ([25], Theorem 2.5.5) implies that, for $\theta \in I'$

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n-1}E_0(X_k - X_{k-1})e^{ik\theta} \rightarrow_n 0$$

\mathbb{P} -a.s. The conclusion follows from these arguments, as explained at the beginning, via (7.20). \square

26.2 Proof of Theorem 16.6

The proof of theorem 16.6 is, as announced at the beginning of this section, an application of the results in [18]. The proof that we will present here depends on the following lemma:

Lemma 18. *In the context on page 89. If*

$$\sum_{k \in \mathbb{N}^*} \frac{\|E_0S_k\|_{\mathbb{P},2}}{k^{3/2}} < \infty, \quad (7.22)$$

then $E_0S_k/\sqrt{n} \rightarrow_n 0$, \mathbb{P} -a.s.

Proof: This is a direct consequence of Theorem 4.7 in [18]: with the notation of that paper, take X_0 in place of f , $\psi = 1$ (the constant function), E_0T in place of T , and use (1.48). \square

Proof of Theorem 16.6: Given $\theta \in I$, let \tilde{T}_θ and \tilde{X}_0 be the extensions of X_0 and T recalled along the “Step 2.” in the proof of Lemma 9 (page 105). Keeping the notation introduced there note that, by an argument analogous to the one leading to the chain of equalities (6.20),

$$\sum_{k \in \mathbb{N}^*} \frac{\|\tilde{E}_0 \tilde{S}_k(\theta)\|_{\lambda \times \mathbb{P}, 2}}{k^{3/2}} = \sum_{k \in \mathbb{N}^*} \frac{\|E_0 S_k(\theta)\|_{\mathbb{P}, 2}}{k^{3/2}}, \quad (7.23)$$

so that, if the last series is convergent, Lemma 18 implies that there exists $\tilde{\Omega}_0 \subset [0, 2\pi) \times \Omega$ with $(\lambda \times \mathbb{P})\tilde{\Omega}_0 = 1$ such that for every $(u, \omega) \in \tilde{\Omega}_0$

$$0 = \lim_n \frac{\tilde{E}_0 \tilde{S}_k(\theta)}{\sqrt{n}}(u, \omega) = \lim_{n \rightarrow \infty} e^{iu} \frac{E_0 S_k(\theta)}{\sqrt{n}}(\omega),$$

and it follows that $E_0 S_k(\theta)/\sqrt{n} \rightarrow_n 0$, \mathbb{P} -a.s. The conclusion follows again via Corollary 16.2. \square

Bibliography

- [1] Andersen, E. S. and Jessen, B. (1948) On the Introduction of Measures in Infinite Product Sets. *Det Kongelige Danske Videnskabernes Selskab Matematik-Fysike Meddelelser*. 25(4).
- [2] Assani, I (2003) Wiener-Wintner Ergodic Theorems. *World Scientific Publishing Co. Pte. Ltd.*
- [3] Barrera, D (2015+16). Quenched Invariance Principles for the Discrete Fourier Transforms of a Stationary Process. *Submitted*.
- [4] Barrera, D. (2015). An Example of non-quenched Convergence in the Conditional CLT for Discrete Fourier Transforms. *ALEA Lat. Am. J. Probab. Math. Stat.* **12** (2), 699–711.
- [5] Barrera, D. and Peligrad, M. (2016) Quenched Limit Theorems for Fourier Transforms and Periodogram. *Bernoulli*. Vol **22**, no 1, 275–301.
- [6] Barrera, D; Peligrad, C and Peligrad, M. On the Functional CLT for Stationary Markov Chains Started at a Point. T appear in *Stoch. Proc. Appl.*
- [7] Bauer, H. (1996) Probability Theory. *De Gruyter Stud. Math.* **23**. De Gruyter. Berlin-New York.
- [8] Bhattacharya, R and Waymire, E.C (2007). A basic course in probability theory. *Universitext*. Springer Science+Business Media, LLC.
- [9] Billingsley, P. (1968) Convergence of Probability Measures, 1st ed. *John Wiley & Sons, Inc.*
- [10] Billingsley, P. (1999) Convergence of Probability Measures, 2nd ed. *Wiley-Intersci. Publ.* John Wiley & Sons, Inc.
- [11] Billingsley, P. (1995) Probability and Measure, 3rd ed. *Wiley-Intersci. Publ.* John Wiley & Sons, Inc.
- [12] Billingsley, P. (1961) Statistical Inference for Markov Processes. *The University of Chicago Press*. Chicago.
- [13] Bourgain, J. (1989) Pointwise Ergodic Theorems for Arithmetic Sets *Publ. Math. Inst. Hautes Etudes Sci.*, no **69**, 5–41.

- [14] Brockwell, P and Davis, R. (2006) Times Series: Theory and Methods, 2nd ed. *Springer Science+Business Media, LLC*.
- [15] Carleson, L. (1966). On convergence and growth of partial sums of Fourier series. *Acta Math.* **116**. 135-157.
- [16] Cohen, J and Conze, J-P (2013). The CLT for Rotated Ergodic Sums and Related Processes. *Discrete Contin. Dynam. Systems.* Vol **33** no 9 September 2013. 3981-4002.
- [17] Cuny, C. (2011) Pointwise Ergodic Theorems with rate and Application to Limit theorems for Stationary Processes, *Stoch. Dyn.* **11**, 135–155.
- [18] Cuny, C and Merlevede, F. (2014). On Martingale Approximations and the Quenched Weak Invariance Principle. *Ann. Probab.* Vol **42**. no 2, 760–793.
- [19] Cuny, C., Merlevede, F. and Peligrad, M. (2013). Law of the iterated logarithm for the periodogram. *Stoch. Proc. Appl.* **123** 4065-4089.
- [20] Cuny, C. and Peligrad, M. (2012). Central Limit Theorem Started at a Point for Stationary Processes and Additive Functionals of Reversible Markov Chains. *J. Theor. Probab.* **25** 171–188.
- [21] Cuny, C. and D. Volný (2013). A quenched invariance principle for stationary processes. *ALEA-Lat. Am. J. Probab. Math. Stat.* **10** 107–115.
- [22] Dedecker J., Merlevède F. and M. Peligrad (2014). A quenched weak invariance principle. *Ann. Inst. H. Poincaré Probab. Statist.* **50** 872-898.
- [23] Dehling, H. Durieu, O. and Volný, D. (2009). New techniques for Empirical Processes of Dependent Data. *Stoch. Proc. Appl.* **119**, 3699-3718.
- [24] Derriennic, Y. and Lin, M. (2001). The Central Limit Theorem for Markov Chains with Normal Transition Operators, started at a Point. *Probab. Theory Relat. Fields.* **119**, 508-528.
- [25] Durrett, R. (2013). Probability: Theory and Examples, 4th ed. *Camb. Ser. Stat. Probab. Math.* Cambridge Univ. Press. Cambridge.
- [26] Eisner, T. Farkas, B. Haase, M. and Nagel, R. (2015) Operator Theoretic Aspects of Ergodic Theory. *Grad. Texts in Math.* **272**. Springer.
- [27] Einsiedler, M. Ward, T. (2011) Ergodic Theory-with a view towards Number Theory. *Grad. Texts in Math.* **259**. Springer.
- [28] Furstenberg, H. (1960) Stationary Processes and Prediction Theory, *Ann. of Math. Stud.* **44**. Princeton Univ. Press, Princeton. NJ.
- [29] Furstenberg, H. (1977) Ergodic Behavior of Diagonal Measures and a Theorem of Szemerédi on Arithmetic Progressions. *J. Analyse. Math.* **31**, 204–256.
- [30] Gordin, M. (1969) The Central Limit Theorem for Stationary Processes. *Soviet. Math Dokl.* 10. n.5, 1174-1176.

- [31] Gordin M. I. and Lifšic, B.A. (1981). A Remark About a Markov Process with Normal Transition Operator. *Third Vilnius Conf. Proba. Stat.* Akad. Nauk Litovsk, (in Russian), Vilnius, **1**, 147-148.
- [32] Grafakos, L. (2008) Classical Fourier Analysis, 2nd ed. *Grad. Texts in Math.* **249**. Springer.
- [33] Hunt, R. I. and Young, W.S (1974). A weighted norm inequality for Fourier series. *Bull. Amer. Math. Soc.* **80**. 274-277.
- [34] Kipnis, C; Varadhan, S.R.S. (1986) Central Limit Theorem for Addictive Functionals of Reversible Markov Processes and Applications to Simple Exclusions, *Commun.Math.Phys.* **104**. 1-19.
- [35] Krengel, U. (1985) Ergodic Theorems, *de Gruyter Stud. Math.* 6. De Gruyter. Berlin-New York.
- [36] Lacey, M. (2004) Carleson's Theorem: Proof, Complement, Variations. *Publ. Mat.* **48**. no 2, 251–307.
- [37] Lifshits, M.A and Peligrad, M (2015) On the Spectral Density of Stationary Processes and Random Fields. *Zapiski Nauchnyh Seminarov POMI, vol. 441*. Probability and Statistics 22 (editors A.N.Borodin, M.A.Lifshits, A.Yu.Zaitsev) 274-286.
- [38] Maxwell, M and Woodroffe, M. (2000) *Central Limit Theorems for Additive Functionals of Markov Chains*, Ann. Probab. **28**, 713–72.
- [39] Merlevéde, C; Peligrad, C and Peligrad, M. (2011) Almost Sure Invariance Principles via Martingale Approximation. *Stoch. Proc. Appl.* **122**. 70–190.
- [40] Neveu, J (1970) Bases Mathématiques du Calcul des Probabilités. 2nd ed. *Masson et Cie*. Paris VIe.
- [41] Peligrad, M. and W. B. Wu (2010). Central limit theorem for Fourier transforms of stationary processes. *Ann. Probab.* **38** 2009-2022.
- [42] Quas, A (2009). Ergodicity and Mixing Properties. *Encyclopedia of Complexity and Systems Science*. 2918-2933.
- [43] Revu, D and Yor, M (2005) Continuous Martingales and Brownian Motion. *Grundlehren Math. Wiss.* Springer. Corrected 3rd printing.
- [44] Schuster, A. (1898) On the Investigation of Hidden Periodicities with Application to a Supposed 26-day Period of Meteorological Phenomena *Terr. Mag.* **3**, 13-41.
- [45] Volný, D. and M. Woodroffe. (2010). An example of non-quenched convergence in the conditional central limit theorem for partial sums of a linear process. *Dependence in analysis, probability and number theory (The Phillipp memorial volume)*, Kendrick Press. 317-323.
- [46] Volný, D and Woodroffe, M (2014). Quenched Central Limit Theorems for Sums of Stationary Processes. *Stat. Probabil. Lett.* **85**, February 2014, 161–167

- [47] Wu, W. B. (2005). Fourier transforms of stationary processes. *Proc. Amer. Math. Soc.* **133** 285-293.
- [48] Wu, W,B and Woodroffe, M. (2004) Martingale Approximations for Sums of Stationary Processes, *Ann. Probab.* **32**. 1674–1690.